Combinatorial Yang–Baxter

Peter J. Cameron
University of St Andrews
(joint with Tatiana Gateva-Ivanova)

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It asserts, that, as maps from $X \times X \times X$ to itself,

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where r^{12} acts as r on the first two coordinates fixing the third, and r^{23} is similarly defined.
Of course, a solution of this equation gives a solution of the "linear" equation over any field F, on setting $V = FX$, the F-vector space with basis X.

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But there is more.
This is by no means a complete survey; I am talking about joint work with Tatiana Gateva-Ivanova, and in particular on my contributions to this work.

What does the CYB equation mean?

We can describe the function r in a different way. If $r(x,y) = (u,v)$, we set $u = f_x(y)$ and $v = g_y(x)$; for each x, the function f_x is a map from X to itself, and similarly for g_y.

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Let $\circ$ be a binary operation on X, and define l_x to be the operation of left translation by x: that is, $l_x(y) = x \circ y$. Then the operation $\circ$ is associative if and only if

$$x \circ y = z \Rightarrow l_x l_y = l_z$$

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The CYB equation has a similar interpretation: it is a kind of "two-dimensional associative law".

Proposition

Let r satisfy the CYB equation on X. If $r(x,y) = (u,v)$, then $f_x f_y = f_u f_v$.

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Proof.

For any $z \in X$,

$$(x,y,z) \xrightarrow{r_{23}} (x, f_y(z), ?) \xrightarrow{r_{12}} (f_x(f_y(z)), ?, ?) \xrightarrow{r_{23}} (f_x(f_y(z)), ?, ?),$$

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□

Thus solutions to CYB are “2-dimensional semigroups”, and of course we want to understand them.

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What I say is true for finite sets. Some of the results extend to infinite sets.

Remarks

Sometimes we write $f_x(y) = {}^x y$ and $g_y(x) = x^y$. We think of X acting on Y (on the left) and Y acting on X (on the right). Non-degeneracy means that these actions are by permutation, so we are in the world of permutation groups.

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Relaxations of the first two conditions have been studied, and some results are known about this.

Two examples

- The function $r(x, y) = (y, x)$ satisfies the Yang–Baxter equation (essentially by the usual proof that the transpositions $(1, 2)$ and $(2, 3)$ in the symmetric group S_3 satisfy the braid relation), and also our three additional conditions. In this case, the functions f_x and g_y are the identity, for all choices of x and y. This is referred to as the trivial solution.

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- The function which swaps $(1, 2)$ with $(3, 1)$, $(1, 3)$ with $(2, 1)$, and $(2, 3)$ with $(3, 2)$ (and fixes all diagonal pairs) is a solution on $X = \{1, 2, 3\}$.

Translation to permutation groups

The functions f and g can be regarded as maps from X into the symmetric group on X. It can be shown that a pair of maps $f, g : X \to \mathrm{Sym}(X)$ arise in this way from a solution if and only if they satisfy the following conditions:

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The second equation shows that g is determined by f, so everything can be expressed in terms of f.
It also shows that the group generated by the maps g_y is contained in the group generated by the maps f_x; by symmetry these groups are equal.

A problem

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I will say more about this later.

Other algebraic structures

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There is another way of producing a group from a solution of the YBE. For any solution r on X, let

$$G(r) = \langle X \mid xy = uv \text{ whenever } r(x, y) = (u, v) \rangle$$

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The algebras are interesting in other contexts: they are quadratic algebras, that is, their relations are homogeneous quadratic expressions in the generators.

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One of our results is:

Theorem

The derived length of $G(r)$ is one greater than the derived length of $\mathcal{G}(r)$.

Retracts and multipermutation level

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It is easy to see that the given solution induces a solution (called a retract) on the set of equivalence classes.
We can repeatedly take retracts; if the corresponding congruences are always non-trivial, eventually we get a solution on a 1-element set. Such a solution is called a multipermutation solution; its level is the number of retractions required to reach the 1-element set.

A multipermutation solution of level 0 is just the trivial solution on a 1-element set. A multipermutation solution of level 1 is the trivial solution on a set of size greater than 1; so the associated permutation group $\mathcal{G}(r)$ is the trivial group, and the group $G(r)$ is (free) abelian.

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For a multipermutation solution, the YB group is soluble with derived length bounded by the multipermutation length; so the derived length of the YB permutation group is bounded by one less than the multipermutation length.

Level and cardinality

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We also have a structure theorem for finite solutions with abelian YB permutation group. They are necessarily multipermutation, with level at most the number of orbits of $\mathcal{G}(r)$. They can be constructed from the solutions corresponding to the orbits by a construction known as strong twisted union. Every finite abelian group is isomorphic to the YB permutation group of such a solution.

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There is surely much more to say about this situation!

Sketch proof of Theorem 2

Let G and G^* be the YB group and permutation group associated with a solution r. We have to show that $\mathrm{dl}(G) = \mathrm{dl}(G^*) + 1$.

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First note that we have a homomorphism from G to G^*, mapping x to f_x. From the definition of the group G we find that the kernel of the homomorphism is abelian, so $\mathrm{dl}(G) \leq \mathrm{dl}(G^*) + 1$. Indeed, the kernel is free abelian.

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Lemma

Let A be free abelian of finite rank, and H a finite group acting faithfully on A. Then $[H, A] = \langle {}^h a - a : a \in A, h \in H \rangle$ is non-zero, and H acts faithfully on $[H, A]$.

Now we prove the theorem. It is known that there is a natural number p such that the subgroup of G generated by the pth powers of the generators is a free abelian group A, and admits a faithful action by G^*. Define A_n inductively by $A_0 = A$ and $A_{n+1} = [G^{(n)}, A_n]$. Using our lemma, if $G^{(n)} \neq 1$, then $A_{n+1} \neq 1$; so $A_l \neq 1$, where $l = \mathrm{dl}(G^*)$. Since $A_l \leq G^{(l)}$, we see that $\mathrm{dl}(G) > \mathrm{dl}(G^*)$, and so by the inequality at the start of this section, we have $\mathrm{dl}(G) = \mathrm{dl}(G^*) + 1$.

Wreath products

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We can construct solutions with arbitrarily large multipermutation level and derived length by the *wreath product* construction.
Let (X_0, r_0) and (Y, r_Y) be solutions. Let X be the disjoint union of $|Y|$ copies of X_0, say $\bigcup_{\alpha \in Y} X_\alpha$, and define a function r on $X \times X$ as follows:

$$\begin{aligned} r(x, x') &= r_\alpha(x, x') \qquad \text{if } x, x' \in X_\alpha \text{ for some } \alpha, \\ r(x, x') &= (x', x) \qquad \text{otherwise.} \end{aligned}$$

Define a map $\sigma : Y \to \mathrm{Sym}(X)$ by the rule that $\sigma(y)$ acts on the copies of X_0 in the same way that f_y acts on Y. Then we can construct a solution on $Z = X \cup Y$: we use the given solution on Y, and the solution just constructed on X, and set $r(y, x) = (\sigma(y)x, y)$ and $r(x, y) = (y, \sigma(y)^{-1}x)$ for $x \in X$, $y \in Y$.

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Proposition

The above construction gives a solution. Its permutation group G_Z^* is the wreath product of $G_{X_0}^*$ and G_Y^*, acting in the usual (imprimitive) way on X and as the top group G_Y^* on Y.
Its multipermutation level is given by
$\mathrm{mpl}(Z) = \mathrm{mpl}(X_0) + \mathrm{mpl}(Y) - 1.$

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The bounds it gives are not best possible; it is possible to bring them down with extra care.

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- Is it true that every finite soluble group $\mathcal{G}$ is a YB permutation group? If so, what is the smallest cardinality of a corresponding solution?
- Is it possible to use the theory of transformation semigroups to extend some of these results to the degenerate case?