Determinantal identities for doubly refined enumerations of Alternating Sign Matrices

Luigi Cantini

LPTM, Université de Cergy-Pontoise (France)

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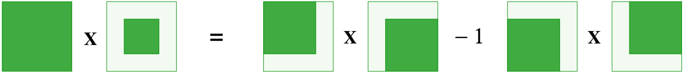
P. Biane, L.C., A. Sportiello arXiv:1110.2404
L.C. : to appear (hopefully soon)

In memory of Alain Lascoux

Alternating Sign Matrices

Alternating Sign Matrices were introduced by Robbins and Rumsey in '83, in their modified version of the Dodgson's condensation algorithm for the evaluation of determinants.

$$\det\ M \det\ M_{1,n}^{1,n} = \det\ M_n^n \det\ M_1^1 - 1 \det\ M_1^n \det\ M_n^1$$



The result was (surprisingly) a Laurent polynomial in entries m_{ij} :

Theorem [Robbins, Rumsey])

$$\det_\lambda M = \sum_{B \in ASM_n} \lambda^{I(B)}(1+\lambda^{-1})^{N(B)} \prod_{i,j} m_{i,j}^{B_{i,j}}$$

$\mathrm{ASM}_n = n \times n$ Alternating Sign Matrices

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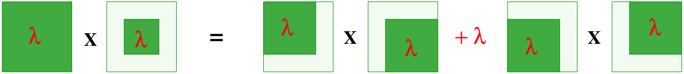

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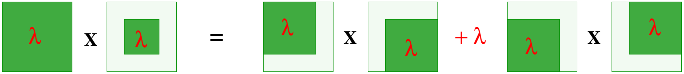

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Alternating Sign Matrices are square $n \times n$ matrices with entries $0, 1, -1$, such that

- signs $+1$ and -1 alternate on each row and each column;
- each row and each column sums to 1.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The number of $n \times n$ ASM has a nice factorized formula

Theorem: ASM enumerations [Zeilberger '96])

$$A(n) = \prod_{j=0..n-1} \frac{(3j+1)!}{(n+j)!}$$

Simpler proof by Kuperberg: use equivalence with 6-vertex (see later)

ASM and Plane Partitions or lozange tilings

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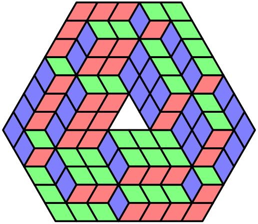
Alternating Sign Matrices are in deep (and somehow mysterious) relations with other remarkable combinatorial objects:

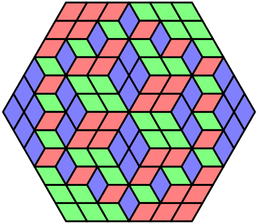
Descending Plane Partitions
They can be seen as lozange covering of an hexagonal region with a central triangular hole and symmetric under a $\frac{2\pi}{3}$ rotations.



ASM and Plane Partitions or lozange tilings

Alternating Sign Matrices are in deep (and somehow mysterious) relations with other remarkable combinatorial objects:

Totally Symmetric Self-Complementary Plane Partitions
They are lozange tilings of a regular hexagon, invariants under the whole the dihedral group of symmetries of the hexagon.



ASM and plane partitions

Combining results from Andrews and Zeilberger it turns out that

Theorem

$$A(n) = DPP(n) = TSSCPP(2n)$$

- These equalities extend to certain weighted and refined enumerations (see later).
- There is no known bijection between these classes of objects. Only partial results: see recent work of Biane and Cheballah for ASM↔TSSCP and Ayyer for ASM↔DPP.

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Weighted enumerations

Beside the simple enumeration it is interesting to introduce certain weights:

- $N(B)$: number of -1 present in the matrix B
- Inversion number

$$I(B) := \sum_{\substack{1\le i<i'\le n\\ 1\le j'\le j\le n}} B_{i,j}B_{i',j'},$$

which generalizes the inversion number of the permutations

Consider the weighted enumeration

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Row-column refined enumerations

From the definining properties of an ASM it follows that on the top/bottom rows and left/rightmost columns there is a single 1

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

We can consider ASM enumerations refined by the position of some of these 1s. The simplest one is

Theorem: Single refined ASM enumerations [Zeilberger '96]

$$A_j(n) = \frac{(n+j-2)!(2n-j-1)!}{(j-1)!(n-j)!(2n-2)!} A(n-1)$$

where j, is the position of the 1 in the leftmost column.

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The most general one will be

$$A_{i_t,i_b,j_\ell,j_r}(n)$$

where we specify the positions i_t, i_b of the top and bottom 1s and j_ℓ, j_r of the leftmost and rightmost 1s

$$\begin{array}{ccc} & 1 \xrightarrow{\quad i_t \quad} n & \\ \begin{array}{c} n \\ j_\ell \uparrow \\ 1 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} & \begin{array}{c} n \\ \uparrow j_r \\ 1 \end{array} \\ & 1 \xrightarrow[\quad i_b \quad]{} n & \end{array}$$

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- Generating function of weighted Doubly Refined (Row-Row) ASM coincide with the generating functions of DPP (properly weighted and refined) [Beherend, Di Francesco, Zinn-Justin]
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Determinant of doubly refined ASM enumerations

Consider the doubly refined enumerations $A^{(RR)}_{i,j}(n)$, $A^{(RC)}_{i,j}(n)$ as matrices

Theorem [P. Biane-L.C.-A. Sportiello, L.C])

$$\det A^{(RR)}_{i,j}(n) = (-A(n))^{n-3}$$

$$\det A^{(RC)}_{i,j}(n) = A^{n-2}(n)$$

- Notice that while it seems natural to look at $\det A^{(RR)}_{i,j}(n|\tau,\nu)$, it doesn't factorize nicely unless τ and ν satisfy some fancy relation.
- The approach developed in [BCS] is not general enough to obtain the second identity.

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Symmetry classes

Most of the symmetry classes of ASM, share the same nice enumerative properties of the simple ASM.

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Vertically Symmetric Alternating Sign Matrices (VSASM) of size $2n+1 \times 2n+1$

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with its fundamental region.

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Off-Diagonal Symmetric Alternating Sign Matrices (OSASM) of size $2n \times 2n$

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It turns out that

Theorem [Kuperberg])

$$A^{VS}(2n+1) = A^{O}(2n) = \prod_{j=0}^{n-1} (3j+2) \frac{(2j+1)!(6j+3)!}{(4j+2)!(4j+3)!}$$

Double refinements for VSASM and OSASM

For VSASM: we call

$$A_{i,j}^{VS}(2n+1)$$

the number of VSASM of size $2n+1$ such that:

- the 1 in the leftmost column in posistion $i-1$
- the next to central column has $j-1$ entries equal to -1.

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- The index j runs from 1 to n
- Since $A_{i,j}^{VS}(2n+1) = A_{2n-i,j}^{VS}(2n+1)$ we can also restrict to $1 \leq i \leq n$ and think at $A_{i,j}^{VS}(2n+1)$ as an $n \times n$ matrix.

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Double refinements for VSASM and OSASM

For OSASM we just consider the row-column refinement

$$A^{OS}_{i,j}(2n)$$

which consists in fixing the positions of the 1s in the leftmost column and in the bottom row

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Determinant of doubly refined VSASM and OSASM enumerations

Theorem [L.C])

$$\det A^{VS}_{i,j}(2n+1) = \left(-A^{VS}(2n-1)\right)^{n-3}$$

$$\det A^{OS}_{i,j}(2n) = \left(-A^{OS}(2n-2)\right)^{3n-6}$$

These two identities and the ones related to the non symmetric ASM will be deduced from two general determinantal identities and not from a case by case analysis.

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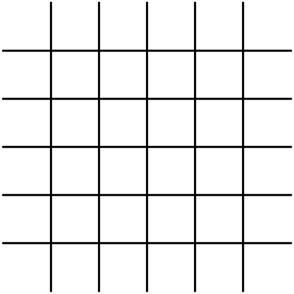
6-Vertex model with domain wall boundary conditions



- Configurations of the 6-vertex model with domain wall boundary conditions are in bijection with ASM.
- To each vertex configuration give a Boltzmann weight which depends on the horizontal and vertical spectral parameter

$$a(z,w) = b(w,z) = qz - q^{-1}w$$
$$c(z,w) = (q^2 - q^{-2})\sqrt{zw}$$

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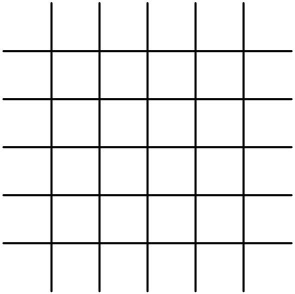

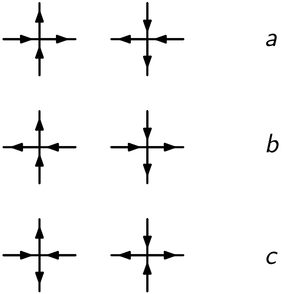

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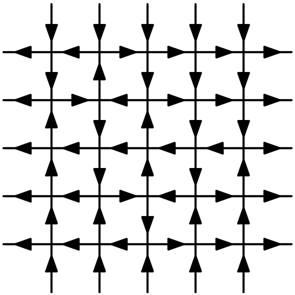

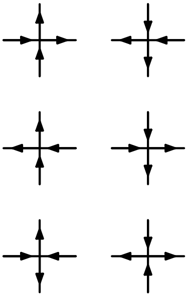

- Configurations of the 6-vertex model with domain wall boundary conditions are in bijection with ASM.
- To each vertex configuration give a Boltzmann weight which depends on the horizontal and vertical spectral parameter

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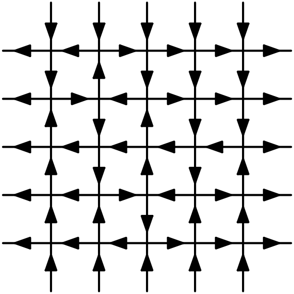

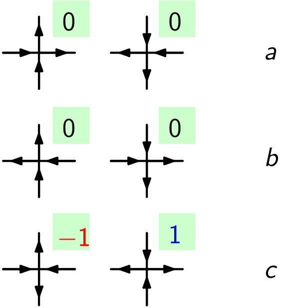

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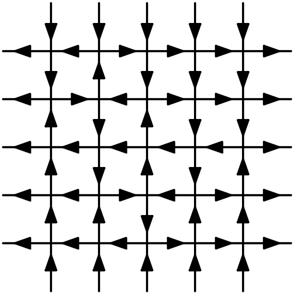

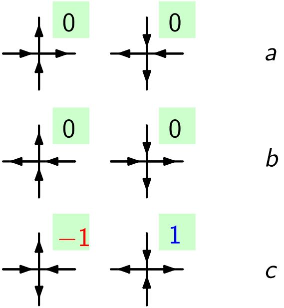

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- For ASM with symmetries, the underlying graphs cover the fundamental domains of the matrix.
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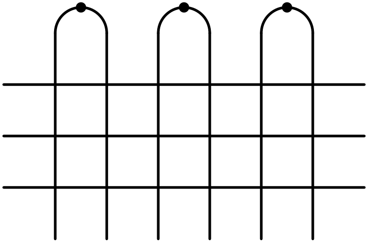
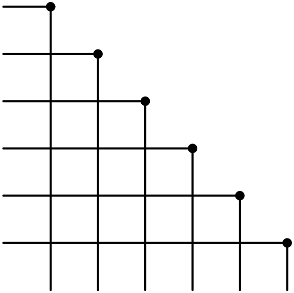


USASM



OSASM

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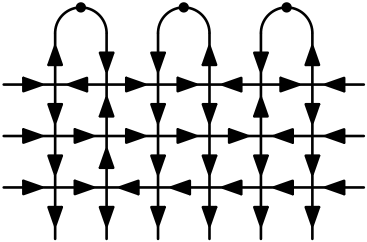
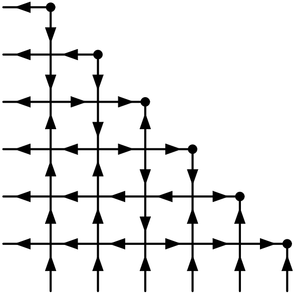


USASM



OSASM

Partition functions

$$Z_n(q;\mathbf{z};\mathbf{w}) = \frac{\prod_{1\le i,j\le n} M(z_i,w_j)}{\Delta(\mathbf{z})\Delta(-\mathbf{w})} \det\left(\frac{c(z_i,w_j)}{M(z_i,w_j)}\right)_{1\le i,j\le n}$$

$$Z^U_{2n}(q;\mathbf{z};\mathbf{w}) = \frac{\prod_{i=1}^n \omega_n(z_i,w_i)\prod_{1\le i,j\le n} M^U(z_i,w_j)}{\Delta(\mathbf{z}+\frac{1}{\mathbf{z}})\Delta(-\mathbf{w}-\frac{1}{\mathbf{w}})} \det\left(\frac{c(z_i,w_j)}{M^U(z_i,w_j)}\right)_{1\le i,j\le n}$$

$$Z^O_{2n}(q;\mathbf{z}) = \frac{\prod_{1\le i<j\le 2n} M^O(z_i,z_j)}{\Delta(\mathbf{z})} \mathrm{Pf}\left(\frac{c(z_i,z_j)(z_i-z_j)}{M^O(z_i,z_j)}\right)_{1\le i,j\le 2n}$$

With

$$\begin{aligned}
M(z,w) &= a(z,w)b(z,w)\\
M^U(z,w) &= a(z,w)b(z,w)a(x^{-1}y^{-1},1)b(x^{-1}y^{-1},1)\\
M^O(z,w) &= a(zw,1)b(zw,1)\\
\omega_n(z,w) &= w^n(\beta-\beta^{-1}w^{-1})(q^2-q^{-2}z^2)
\end{aligned}$$

Partition functions and generating functions

The generating function of row-row doubly refined ASM

$$\mathcal{A}_n^{(R,R)}(\tau,\lambda,x,y) = \sum_{1\leq i,j\leq n} A_{i,j}^{(R,R)}(n|\tau,\lambda)\, x^{i-1} y^{j-1}$$

can be read from the partition function of the 6-vertex models upon specialization of some of the spectral parameters

$$\mathcal{A}_n(\tau,\lambda,x,y) = \left(\frac{b(1,t)a(1,t)}{b(z_1,t)a(z_n,t)}\right)^{n-1} \frac{Z_n(q,z_1,\mathbf{1},z_n;t\mathbf{1})}{c(z_1,t)c(z_n,t)c(1,t)^{n-2}b(1,t)^{n^2-n}}$$

with

$$\tau = \frac{c^2(1,t)}{b^2(1,t)}, \qquad \lambda = \frac{a^2(1,t)}{b^2(1,t)}$$

$$x = \frac{a(z_1,t)b(1,t)}{a(1,t)b(z_1,t)}, \qquad y = \frac{b(z_n,t)a(1,t)}{b(1,t)a(z_n,t)}$$

Modified partition functions and generating functions

In order to extract the generating function of row-column doubly refined ASM from the partition function one need to discard the contribution from the ASMs in which the 1 is in the corner

$$\mathcal{A}_n^{(R,C)}(\tau,\lambda,x,y) = \sum_{1\le i,j\le n} A_{i,j}^{(R,C)}(n|\tau,\lambda)\, x^{i-1}y^{j-1}$$

$$= A(n-1|\tau,\lambda) + xy\tilde{\mathcal{A}}_n^{(R,C)}(\tau,\lambda,x,y)$$

$\tilde{\mathcal{A}}_n^{(R,C)}(\tau,\lambda,x,y)$ is directly related to a partition restricted to the 6-vertex configurations with a vertex of type "a" in the corner

$$Z_n(q;\mathbf{z};\mathbf{w}) = Z_n^c(q;z_1;\mathbf{z};w_1;\mathbf{w}) + a(z_1,w_1)Z_n^{(R,C)}(q;z_1;\mathbf{z};w_1;\mathbf{w})$$

$$\tilde{\mathcal{A}}_n^{(R,C)}(\tau,\lambda,x,y) \propto Z_n^{(R,C)}(q;z_1;\mathbf{1};w_1;t\mathbf{1})$$

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Partition functions: combinatorial point

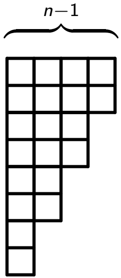
At the combinatorial point $q = e^{i\pi/3}$ the partition functions become symmetric in the joint set of variables $\mathbf{z}, \mathbf{w}$, and equal to a (Symplectic) Schur polynomial

$$Z_n(q = e^{\pi i/3}; \mathbf{z}; \mathbf{w}) = S_{\lambda(n)}(\mathbf{z}, \mathbf{w})$$

$$\frac{Z^U_{2n}(q = e^{\pi i/3}; \mathbf{z}; \mathbf{w})}{\prod_{i=1}^n w_i^n \omega(z_i, w_i)} = Z^O_{2n}(q = e^{\pi i/3}; \mathbf{z}) = S^{Sympl}_{\lambda(n)}(\mathbf{z}, \mathbf{w})$$

corresponding to the double stair Young diagram

$\lambda(n) =$



Generalized Gaudin polynomials

To a pair of polynomials in two variables $P(x,y), Q(x,y)$ we associate a family of polynomials that we call

Generalized Gaudin Polynomials

$$\mathcal{G}_{n,P,Q}(\mathbf{z};\mathbf{w}) = \frac{\prod_{1\le i,j\le n} P(z_i,w_j)}{\Delta(\mathbf{z})\Delta(\mathbf{w})} \det\left(\frac{Q(z_i,w_j)}{P(z_i,w_j)}\right)_{1\le i,j\le n}$$

- They are polynomials symmetric in the variables $\mathbf{z}$ and $\mathbf{w}$
- Natural generalization of 6W-DWBC partition functions
- If $s_{x/y} = \deg_{x/y} P(x,y)$ and $r_{x/y} = \deg_{x/y} Q(x,y)$ then

$$\deg_{z_i/w_j} \mathcal{G}_{n,P,Q}(\mathbf{z};\mathbf{w}) \le (s_{x/y}-1)(n-1) + r_{x/y}$$

- They satisfy nice relations when specialized at $(\bar{x},\bar{y})$, a zero of $P(x,y)$. For $Q(x,y)=1$ these recursion uniquely determine $\mathcal{G}_{n,P,Q}(\mathbf{z};\mathbf{w})$.

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A second family of Gaudin polynomials can be introduced if we suppose that the $P(x, y)$ is divided by a degree 1 polynomial $p_\alpha(x, y)$

$$P(x, y) = p_\alpha(x, y)\tilde{P}(x, y) \quad \text{with} \quad p_\alpha(x, y) = \sum_{i,j=0}^{1} p_{\alpha,i,j} x^i y^j$$

First introduce the polynomials

$$\tilde{\mathcal{G}}^{(\alpha)}_{n,P,Q}(x; \mathbf{z}; y; \mathbf{w}) = Q(x, y) \left(\det p_\alpha\right)^{n-1} \prod_{j=1}^{n-1} \tilde{P}(x, w_j)\tilde{P}(z_j, y)\mathcal{G}_{n-1,P,Q}(\mathbf{z}; \mathbf{w}),$$

where $\det p_\alpha$ is the determinant of matrix of coefficients of $p_\alpha(x, y)$.

Modified Gaudin polynomials)

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Partition functions as Gaudin polynomials

Apart for some trivial factors the partition functions of the 6−vertex model on graphs corresponding to ASM, USASM (and OSASM at the combinatorial point) are Gaudin polynomials

$$Z_n(q;\mathbf{z};\mathbf{w}) \propto \mathcal{G}_{n,M,1}(\mathbf{z};\mathbf{w})$$
$$Z^U_{2n}(q;\mathbf{z};\mathbf{w}) \propto \mathcal{G}_{n,M+(q^2-q^{-2})^2,1}(\mathbf{z}+\mathbf{z}^{-1};\mathbf{w}+\mathbf{w}^{-1})$$

$$Z^{(R,C)}_n(q;x;\mathbf{z};y;\mathbf{w}) \propto \mathcal{G}^{(\alpha)}_{n,M,1}(\mathbf{z};\mathbf{w})$$
$$Z^{OSASM,(R,C)}_{2n+1}(q=e^{i\pi/3};x;\mathbf{z};y;\mathbf{w}) \propto \mathcal{G}^{(\alpha)}_{n,\tilde{M}+(q^2-q^{-2})^2,1}(x;\mathbf{z};y;\mathbf{w})|_{q=e^{i\pi/3}}$$

with

$$M(x,y) = -a(x,y)b(x,y) = x^2+y^2-(q^2+q^{-2})xy$$
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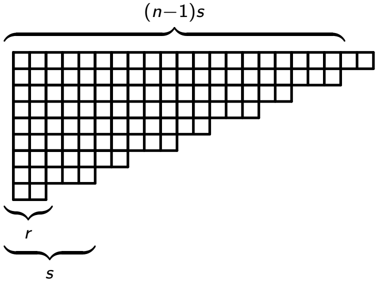
Gaudin Polynomials and Schur functions

Using some identities due to Okada one finds

$$\mathcal{G}_{n,U_{s+2},U_{r+1}}(\mathbf{z};\mathbf{w}) = S_{\lambda(n,s,r)}(\mathbf{z},\mathbf{w})$$
$$\mathcal{G}_{n,\tilde{U}_{s+2},\tilde{U}_{r+1}}(\mathbf{z};\mathbf{w}) = S^{Sympl}_{\lambda(n,s,r)}(\mathbf{z},\mathbf{w})$$

with $U_m(x,y) = \frac{x^m - y^m}{x-y}$ the Chebyshev polynomial of second type, $\tilde{U}_m(x,y) = U_m(x,y)U_m(xy,1)$, $r \leq s$ and

$\lambda(n,s,r) =$



- The Gaudin polynomials with $Q = 1$ and $P(x, y) = \prod_{\alpha=0}^{s-1}(x - t^{\alpha} y)$ have been studied by Lascoux and proven to be given by determinant of certain Schur functions.
- The Gaudin polynomial corresponding to the 6-V partition function ($P(x, y) = M(x, y)$ is diagonal in the basis of MacDonald polynomials ($t \leftrightarrow q$) [Kirillov-Noumi, Warnaar]

$$\prod_{1 \leq i,j \leq} \frac{(qx_i y_j; t)_\infty}{(x_i y_j; t)_\infty} \mathcal{G}_{n,M,1}(\mathbf{z}; \mathbf{w}) = \sum_{\lambda} b_\lambda(t, q) g_\lambda(q; t, q) P_\lambda(x; t, q) P_\lambda(y; t, q)$$

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Determinants of Gaudin Polynomials

Call $s = \min\{s_x, s_y\}$

Theorem

Let $\{x_i, y_i\}_{1 \le i \le m}$ be indeterminates. The polynomial

$$\det \left(\mathcal{G}_{n,P,Q}(x_i, \mathbf{z}; y_i, \mathbf{w}) \right)_{1 \le i,j \le m}$$

is divided by $\mathcal{G}^{m-s-1}_{n-1,P,Q}(\mathbf{z}; \mathbf{w})$.

If $P(x,y)$ is divided by $p_\alpha(x,y)$, then

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is divided by $\mathcal{G}^{m-s}_{n-1,P,Q}(\mathbf{z}; \mathbf{w})$.

The origin of these factors traces back to a classical determinantal identity first discovered by Sylvester

Determinants of Gaudin Polynomials

Call $s = \min\{s_x, s_y\}$

Theorem

Let $\{x_i, y_i\}_{1\le i\le m}$ be indeterminates. The polynomial

$$\det\left(\mathcal{G}_{n,P,Q}(x_i, \mathbf{z}; y_i, \mathbf{w})\right)_{1\le i,j\le m}$$

is divided by $\mathcal{G}^{m-s-1}_{n-1,P,Q}(\mathbf{z};\mathbf{w})$.

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Sylvester identity

Consider a $(n+r)\times(n+r)$ matrix A with a $n \times n$ minor B

$$A = \left(\begin{array}{cc} & \\ \underbrace{}_{r} & \underbrace{B}_{n} \end{array}\right) \left.\vphantom{\begin{array}{c} B \\ B \end{array}}\right\} n$$

For $1 \leq i, j \leq r$, consider the minor $A(i,j)$ formed by B, the ith column and the jth row, and denote its determinant by $a_{i,j} = \det A(i,j)$.

Theorem [Sylvester]

$$\det a = \det A (\det B)^{r-1}$$

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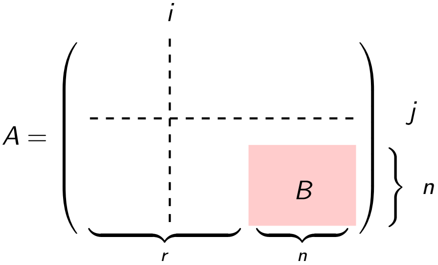

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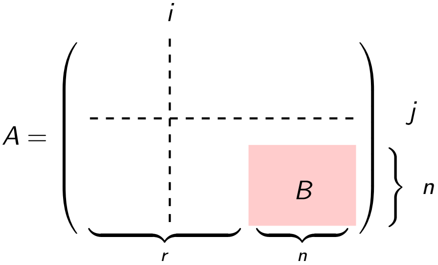

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Determinants of Gaudin Polynomials

Assume now that $s_x = s_y = s$ and $r_x = r_y = r$ and call

$$\mathcal{D}_{n,P,Q}(\mathbf{z};\mathbf{w}) := \frac{\det\left(\mathcal{G}_{n,P,Q}(\mathbf{z},x_i;\mathbf{w},y_j)\right)_{1\le i,j\le (s-1)(n-1)+r+1}}{\Delta(\mathbf{x})\Delta(\mathbf{y})},$$

$$\mathcal{D}^{(\beta)}_{n,P,Q}(\mathbf{z};\mathbf{w}) := \frac{\det\left(\mathcal{G}^{(\beta)}_{n,P,Q}(x_i;\mathbf{z};y_j;\mathbf{w})\right)_{1\le i,j\le (s-1)(n-1)+r}}{\Delta(\mathbf{x})\Delta(\mathbf{y})}.$$

We can provide explicit formulae for these functions.

For $1 \le \alpha \le s$, call $\phi_\alpha(y)$ the zeros of $P(x,y)$ as a polynomial in x, $\psi_\alpha(x)$ the zeros of $P(x,y)$ as a polynomial in y and write

$$P(x,y) = P_1(x)\prod_{\alpha=1}^{s}(y-\phi_\alpha(x)) = P_2(y)\prod_{\alpha=1}^{s}(x-\psi_\alpha(y))$$

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Determinants of Gaudin Polynomials

Theorem

$$\mathcal{D}_{n,P,Q}(\mathbf{z};\mathbf{w}) = k_{n,P,Q}\left(\prod_{1\le i<j\le n-1} G^{(1)}_{n,P}(z_i,z_j)G^{(2)}_{n,P}(w_i,w_j)\right)$$

$$\left(\prod_{j=1}^{n-1}\tilde{G}^{(1)}_{n,P,Q}(z_j)\tilde{G}^{(2)}_{n,P,Q}(w_j)\right)\left(\prod_{1\le i,j\le n-1} P^{s-1}(z_i,w_j)\right)\mathcal{G}^{m-s-1}_{n-1,P,Q}(\mathbf{z};\mathbf{w})$$

$$\mathcal{D}^{(\beta)}_{n,P,Q}(\mathbf{z};\mathbf{w}) = k^{(\beta)}_{n,P,Q}(\mathbf{z},\mathbf{w})\left(\prod_{1\le i<j\le n-1} G^{(1)}_{n,\tilde{P}}(z_i,z_j)G^{(2)}_{n,\tilde{P}}(w_i,w_j)\right)$$

$$\left(\prod_{j=1}^{n-1}\tilde{G}^{(1)}_{n,\tilde{P},Q}(z_j)\tilde{G}^{(2)}_{n,\tilde{P},Q}(w_j)\right)\left(\prod_{1\le i,j\le n-1} p_\beta(z_i,w_j)P^{s-2}(z_i,w_j)\right)\mathcal{G}^{m-s-1}_{n-1,P,Q}(\mathbf{z};\mathbf{w})$$

with $m=(s-1)(n-1)+r+1$.

$$G^{(1)}_{n,P}(z_i,z_j) = \frac{P_1^s(z_i)P_1^s(z_j)\prod_{1\leq\alpha,\beta\leq s}\psi_\alpha(z_i)-\psi_\beta(z_j)}{(z_i-z_j)^s}$$

$$G^{(2)}_{n,P}(w_i,w_j) = \frac{P_2^s(w_i)P_2^s(w_j)\prod_{1\leq\alpha,\beta\leq s}\phi_\alpha(w_i)-\phi_\beta(w_j)}{(w_i-w_j)^s}$$

$$\tilde{G}^{(1)}_{n,P,Q}(z) = P_1^r(z))\prod_{\alpha=1}^{s} Q(z,\phi_\alpha(z))$$

$$\tilde{G}^{(2)}_{n,P,Q}(z) = P_2^r(z))\prod_{\alpha=1}^{s} Q(\psi_\alpha(w),w)$$

These are polynomials of zs and ws.

$$k_{n,P,Q} = (-1)^{s(n-1)}\det Q_{i,j}\left(\det P_{i,j}\right)^{n-1}$$

$k^{(\beta)}_{n,P,Q}(\mathbf{z},\mathbf{w})$ is a polynomial of degree r in z_i and w_j, that I cannot pin down, unless $Q(x,y)=1$.

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