`Workshop on Combinatorial Physics'
Cardiff, 17-19 December 2013

The Arctic Circle re-revisited

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Joint work with:
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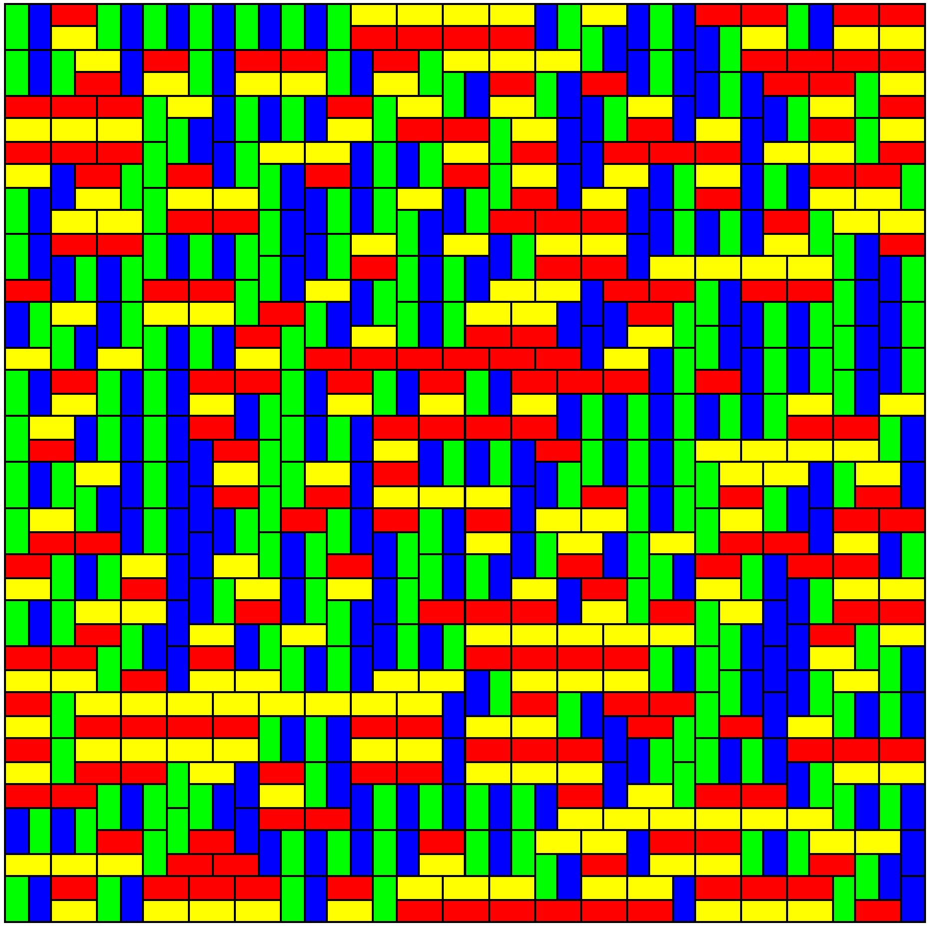
Domino tiling of a square

domino:= 2×1 tile



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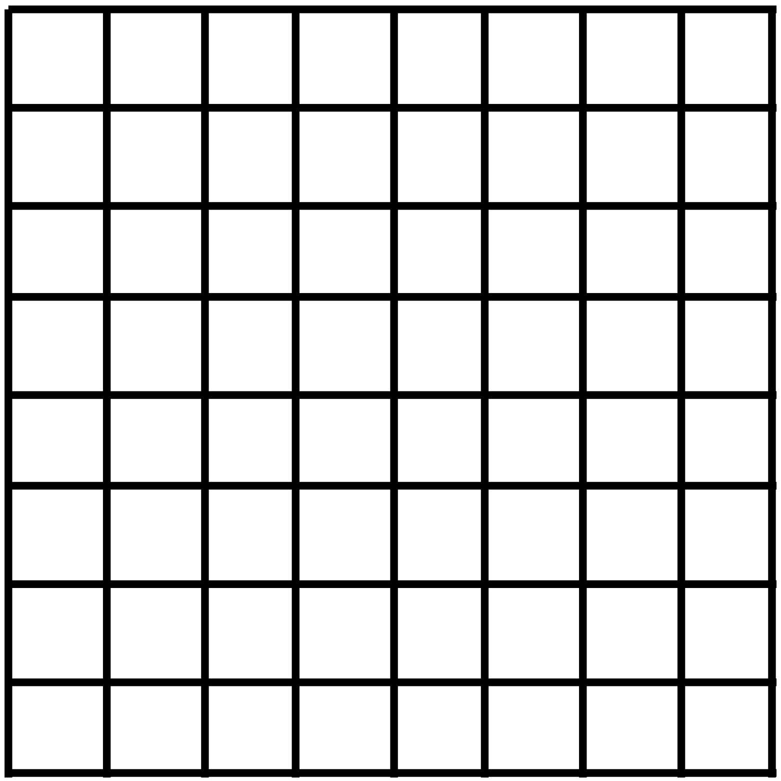
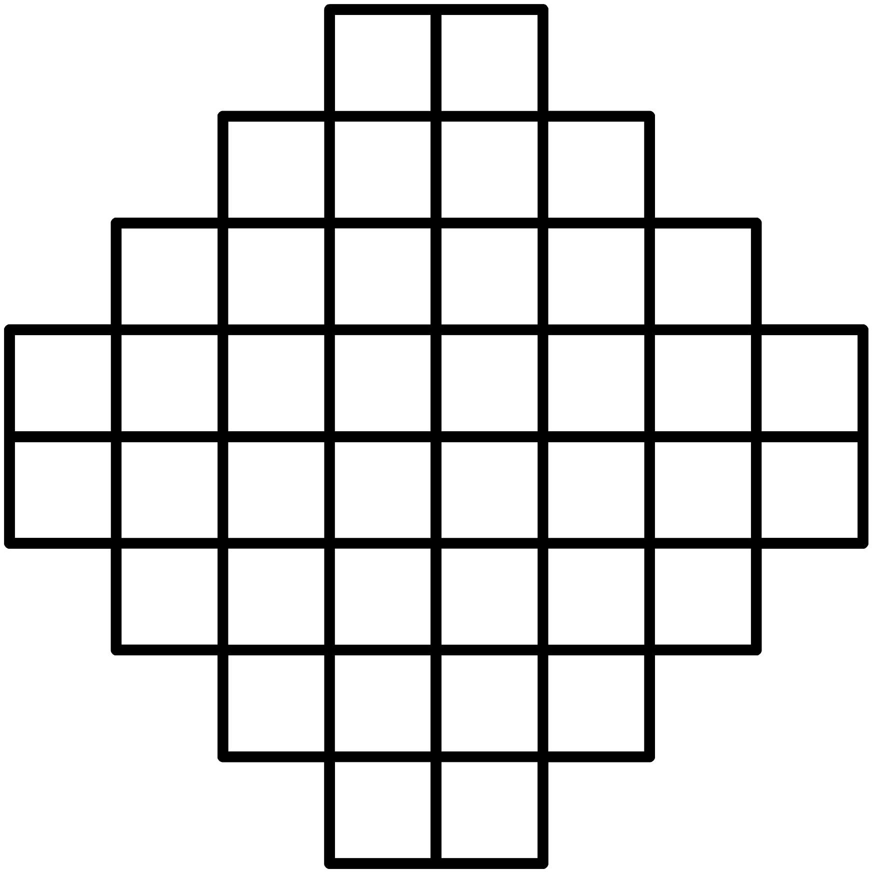


$2N \times 2N$ Square



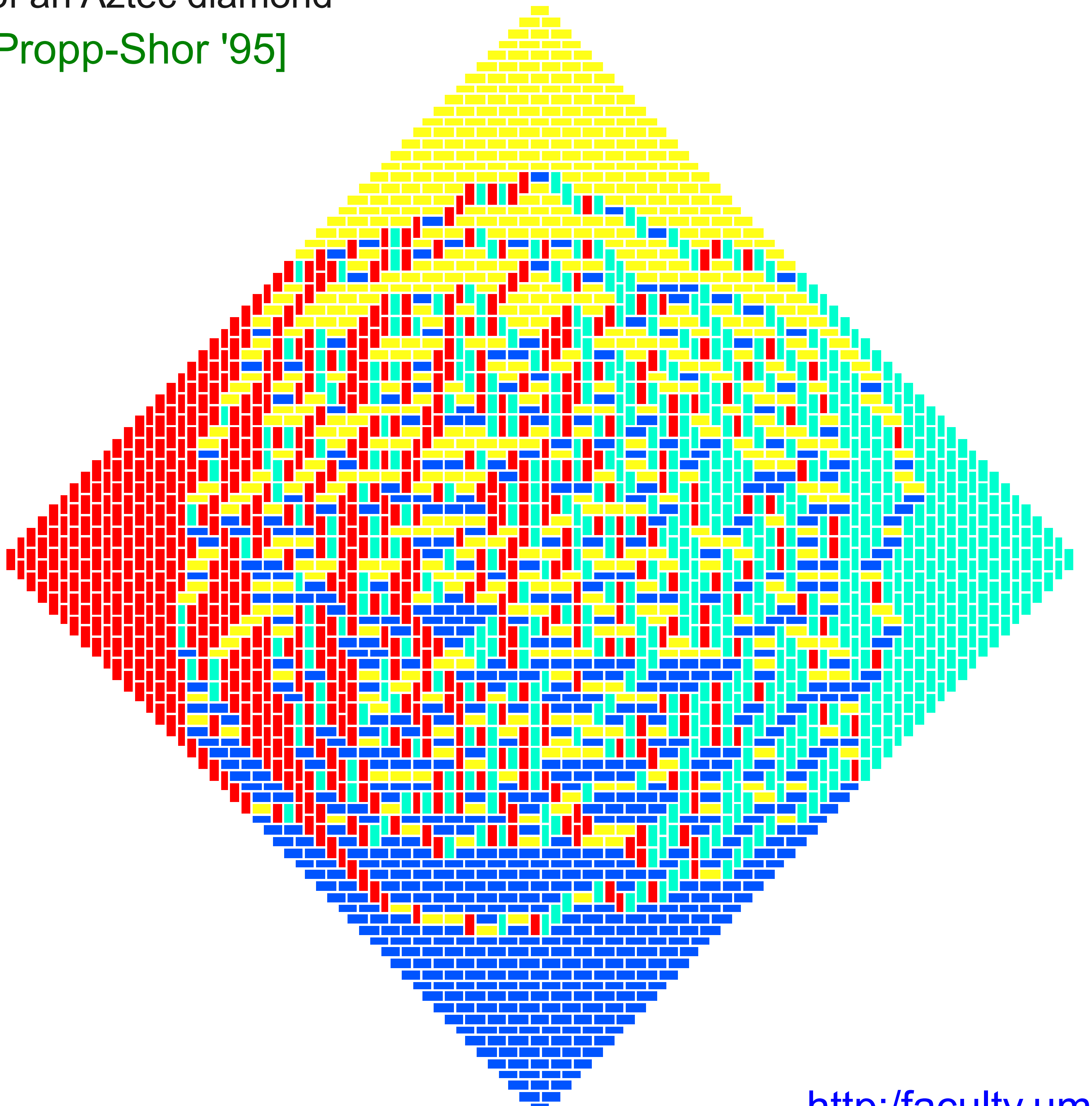
Aztec Diamond of Order N

$(N = 4)$

Domino tiling of an Aztec diamond

[Jockush-Propp-Shor '95]

$N = 64$

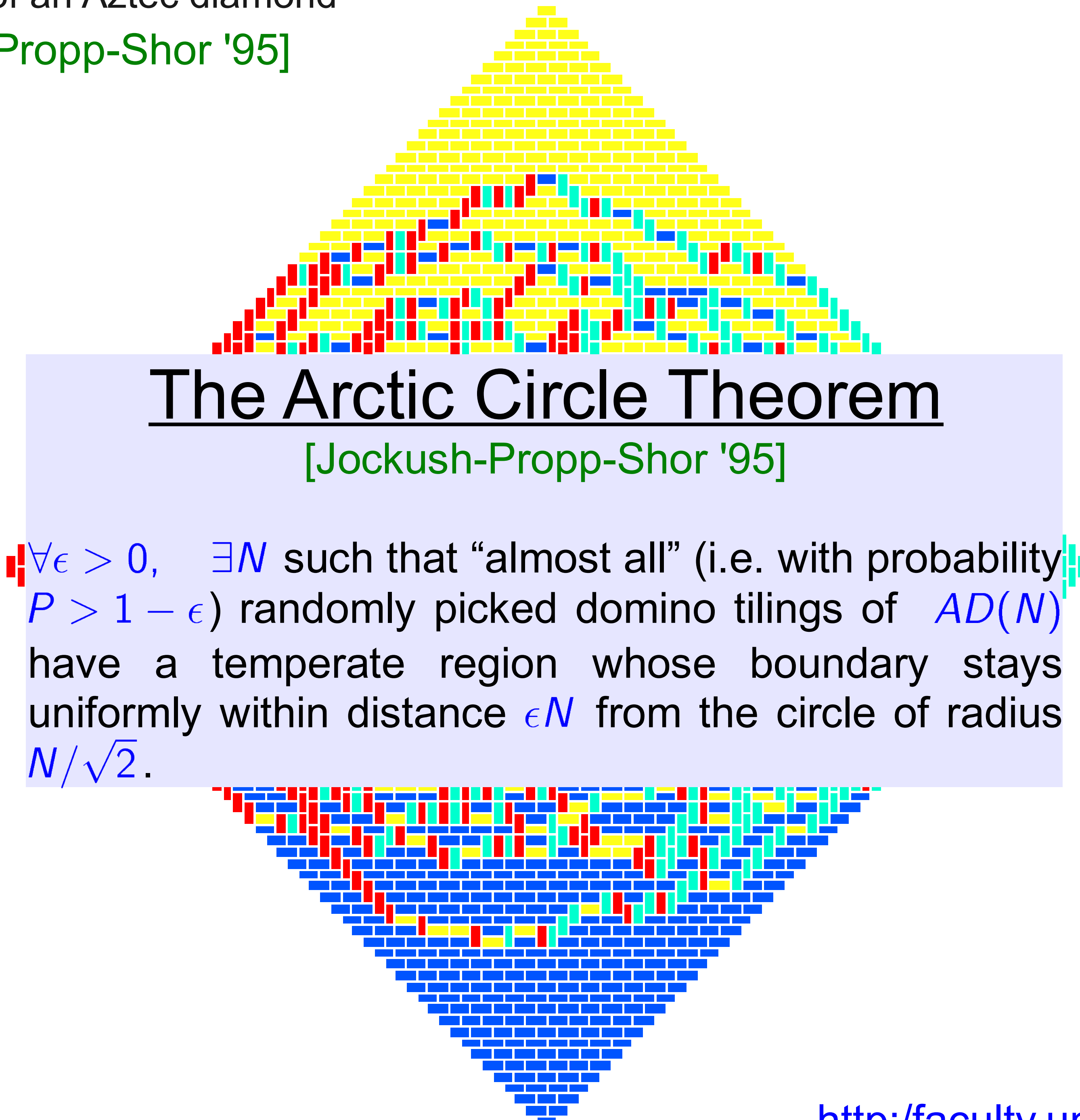


http:/faculty.uml.edu/jpropp

Domino tiling of an Aztec diamond

[Jockush-Propp-Shor '95]

$N = 64$

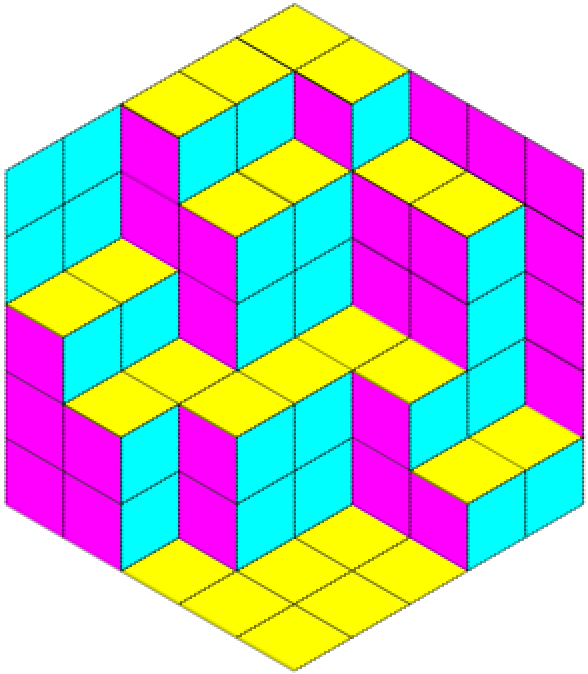


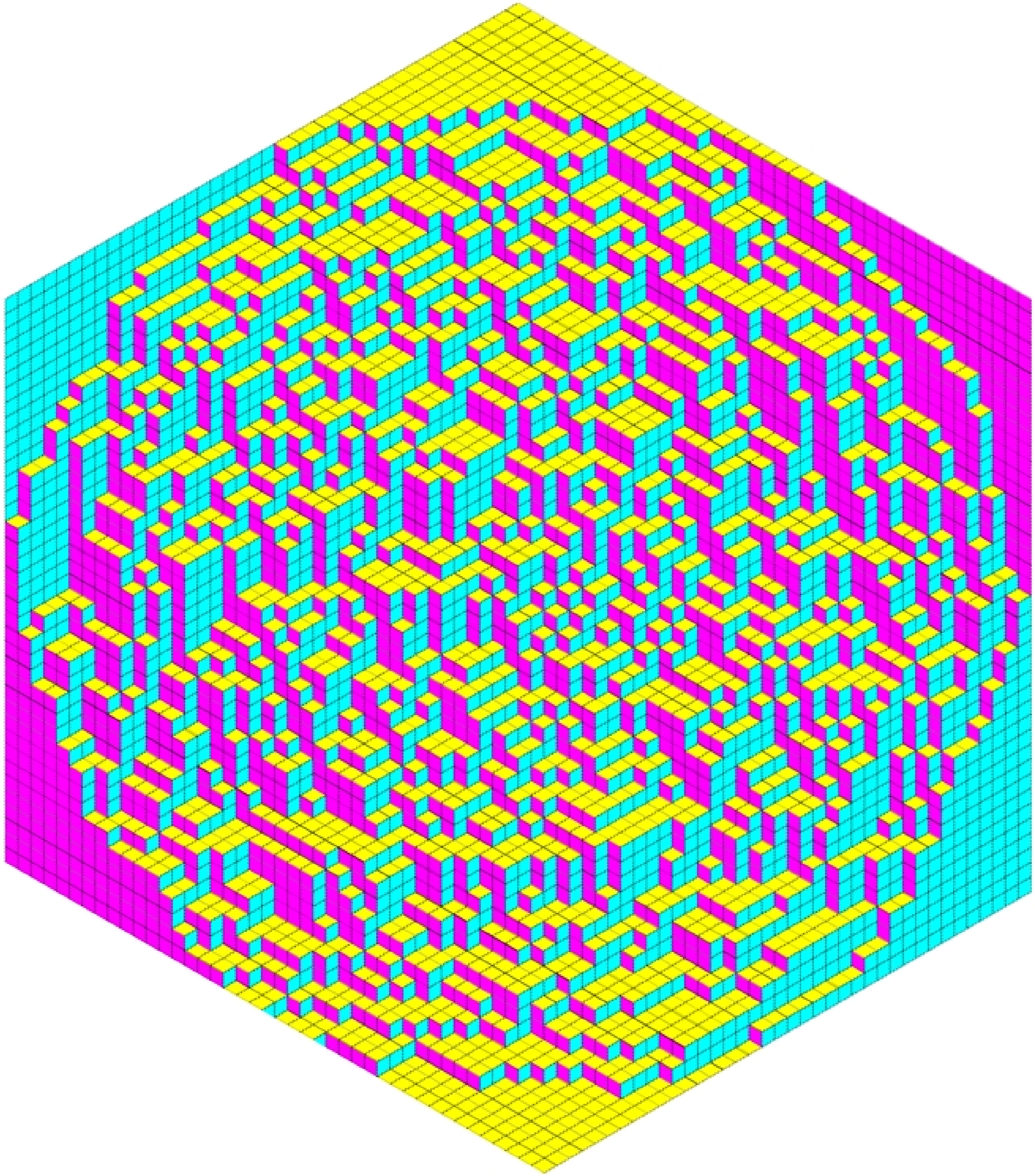
The Arctic Circle Theorem

[Jockush-Propp-Shor '95]

$\forall \epsilon > 0, \quad \exists N$ such that "almost all" (i.e. with probability $P > 1 - \epsilon$) randomly picked domino tilings of $AD(N)$ have a temperate region whose boundary stays uniformly within distance ϵN from the circle of radius $N/\sqrt{2}$.

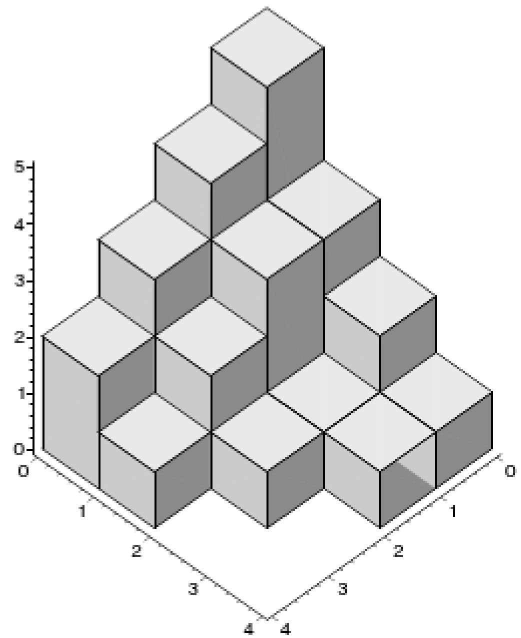
http:/faculty.uml.edu/jpropp

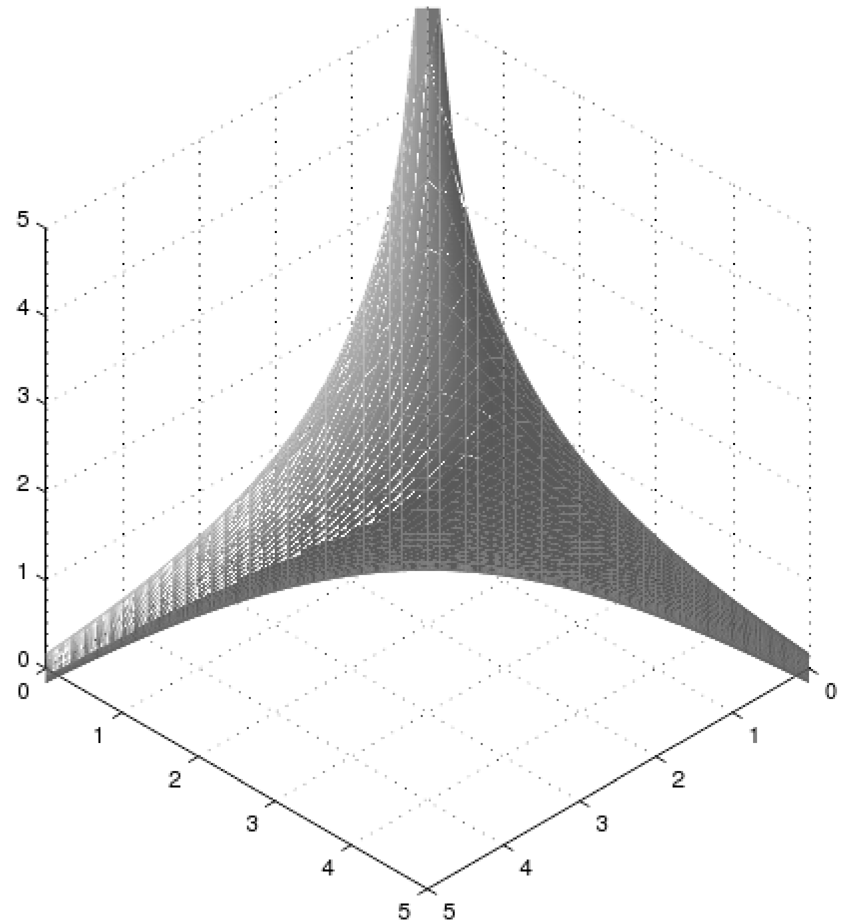




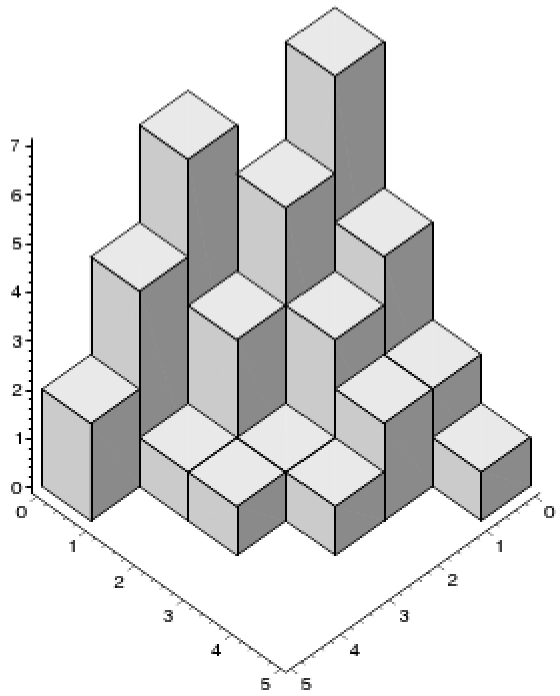
Rhombi tilings of an hexagon
(a.k.a. Boxed plane partitions)

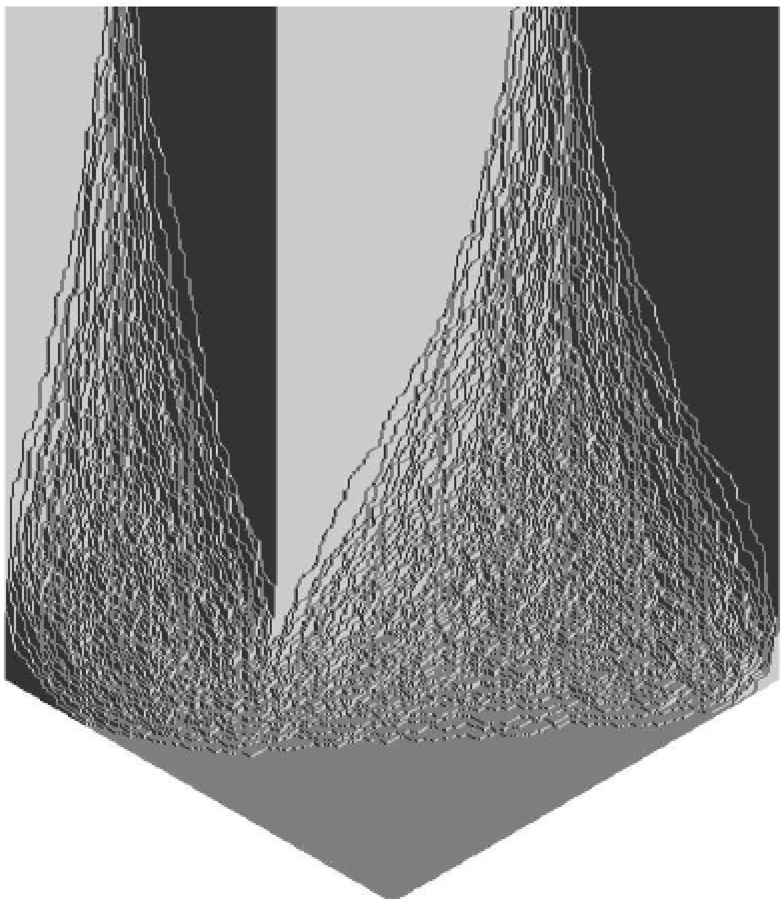
[Cohn-Larsen-Propp'98]





- Corner melting of a crystal [Ferrari-Spohn '02]
- Plane partitions [Cerf-Kenyon'01][Okounkov-Reshetikhin'01]



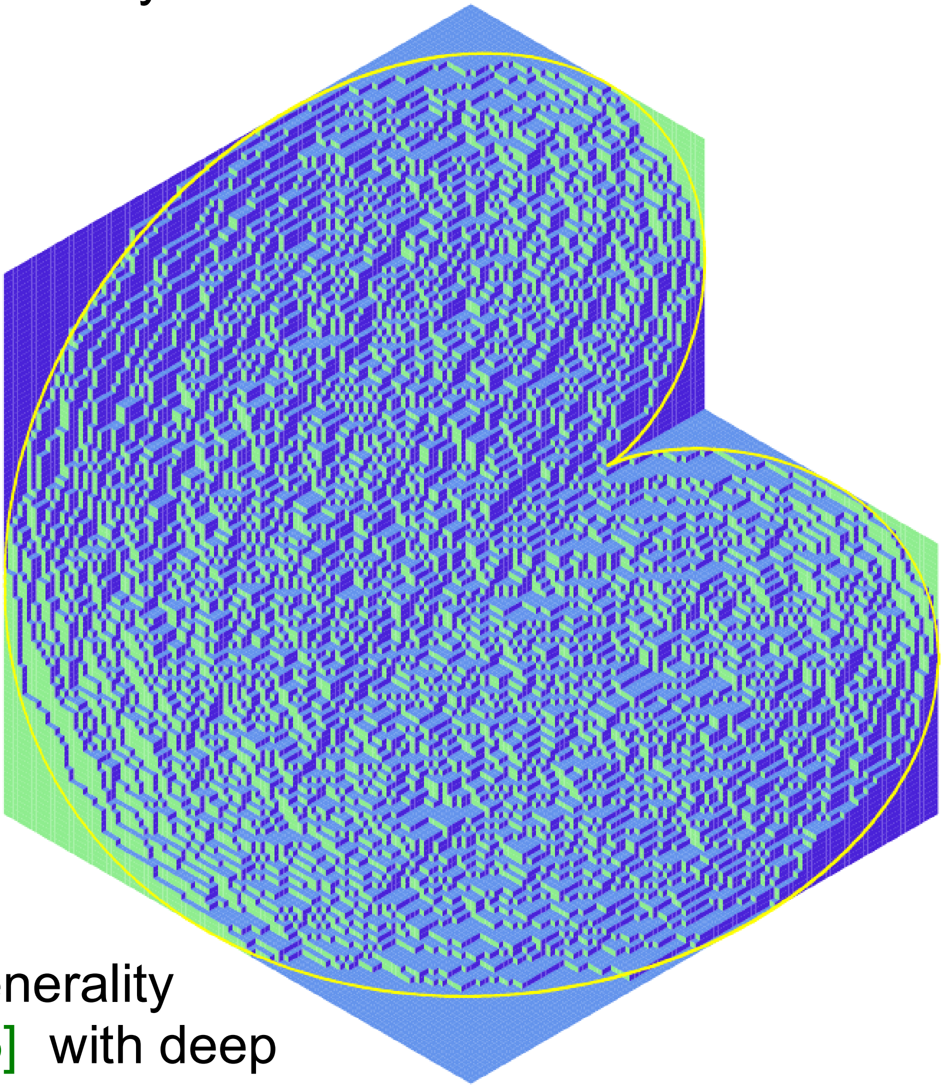


- Skewed plane partitions [Okounkov-Reshetikhin '05-'07]
  [Boutillier-Mkrtchyan-Reshetikhin-Tingley '12]

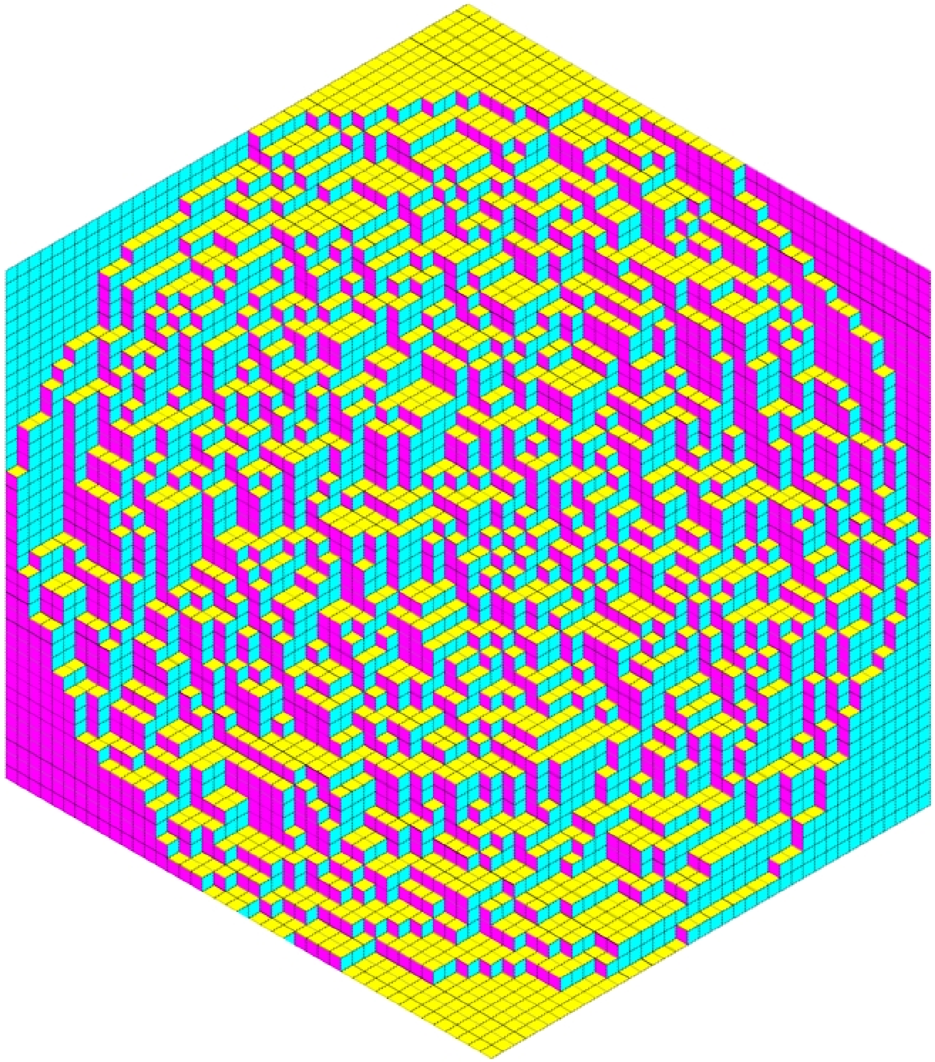
Previously shown models of

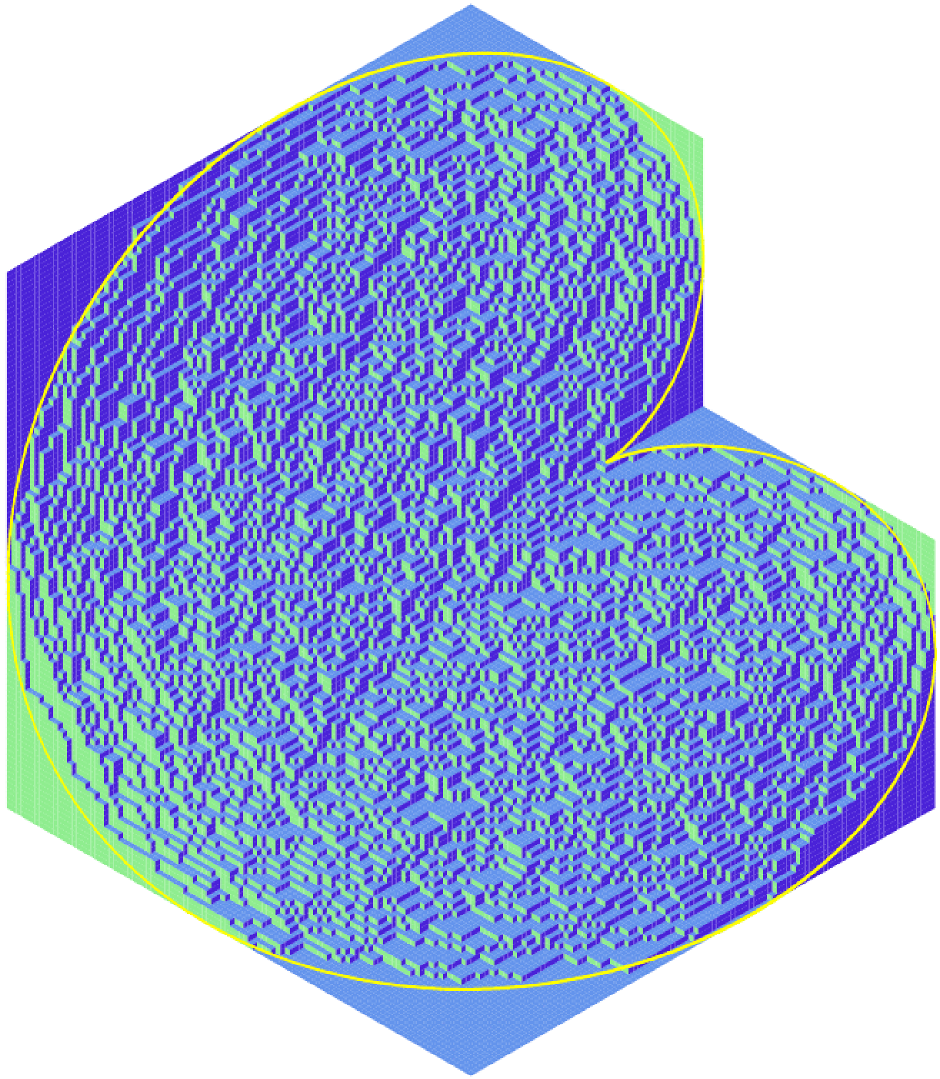
- domino tilings;
- rhombi tilings;
- plane partitions; boxed plane partitions; skewed plane partitions ...

are all avatars of the same model, `dimer covering of regular planar bipartite lattices', exhibiting emergence of phase separation, limit shapes, frozen boundaries /arctic curves, and fluctuations governed by Random Matrix models.

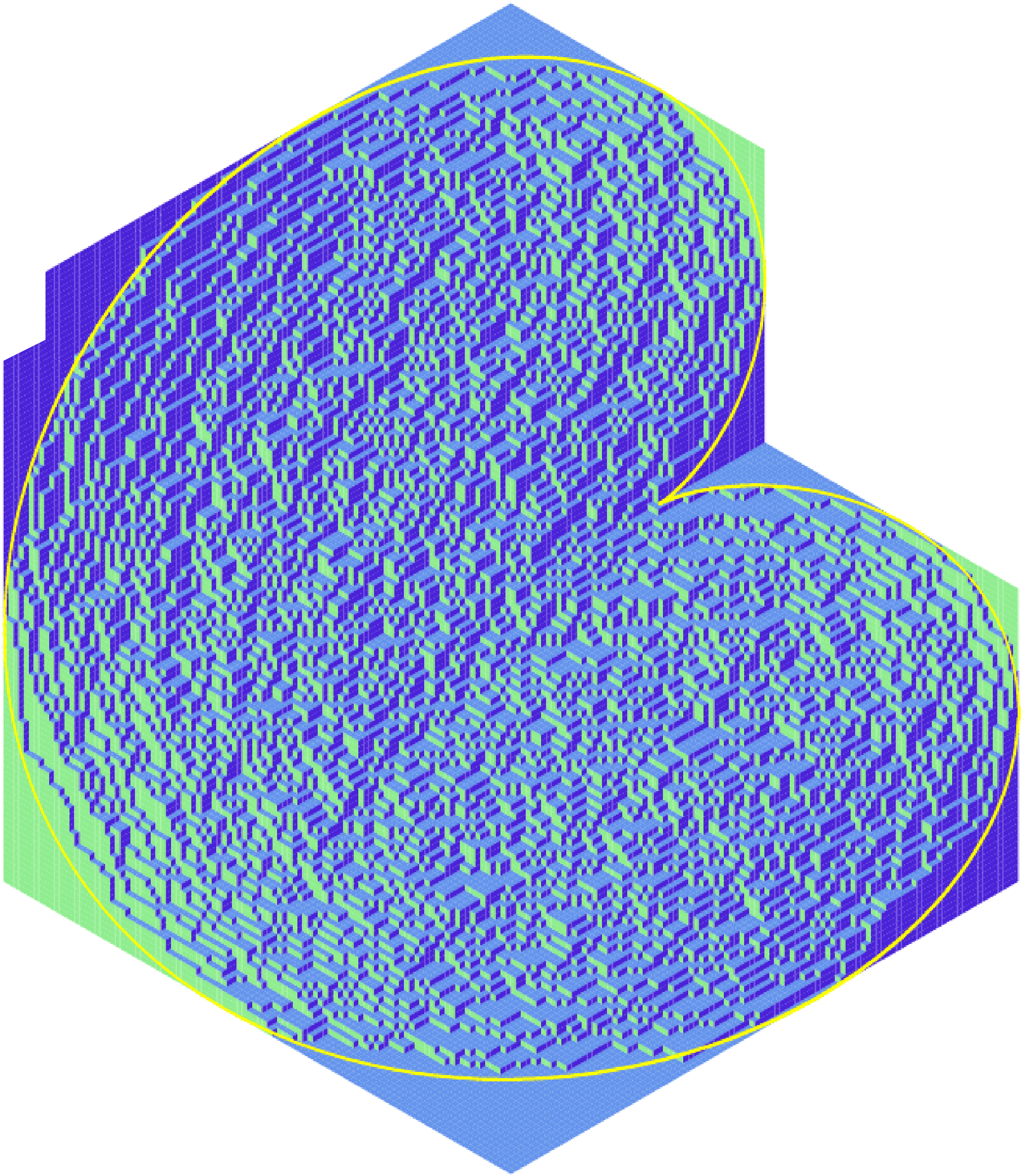


The model has been solved in full generality [Kenyon, Sheffield, Okounkov, '03-'05] with deep implications in algebraic geometry and algebraic combinatorics.

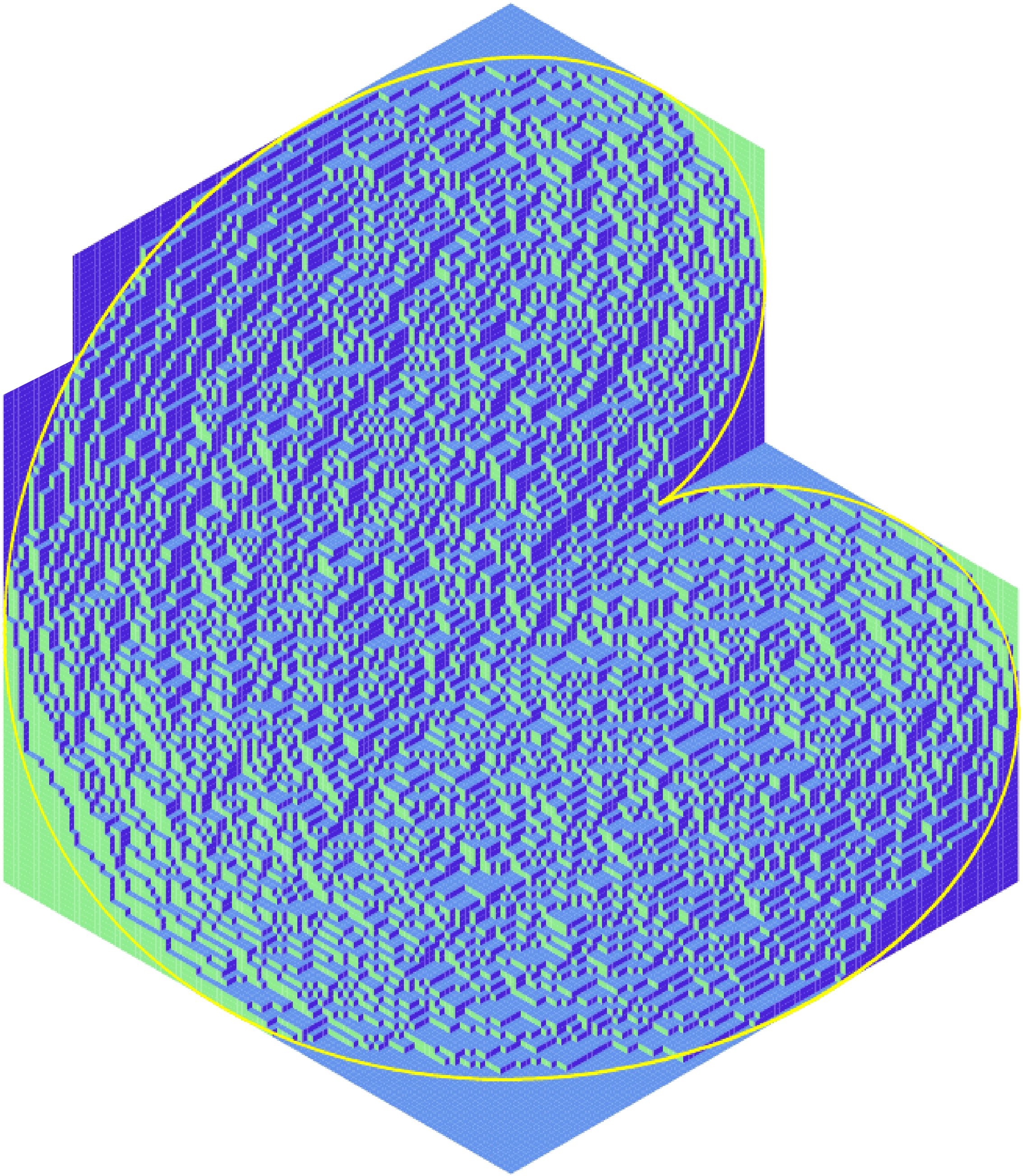




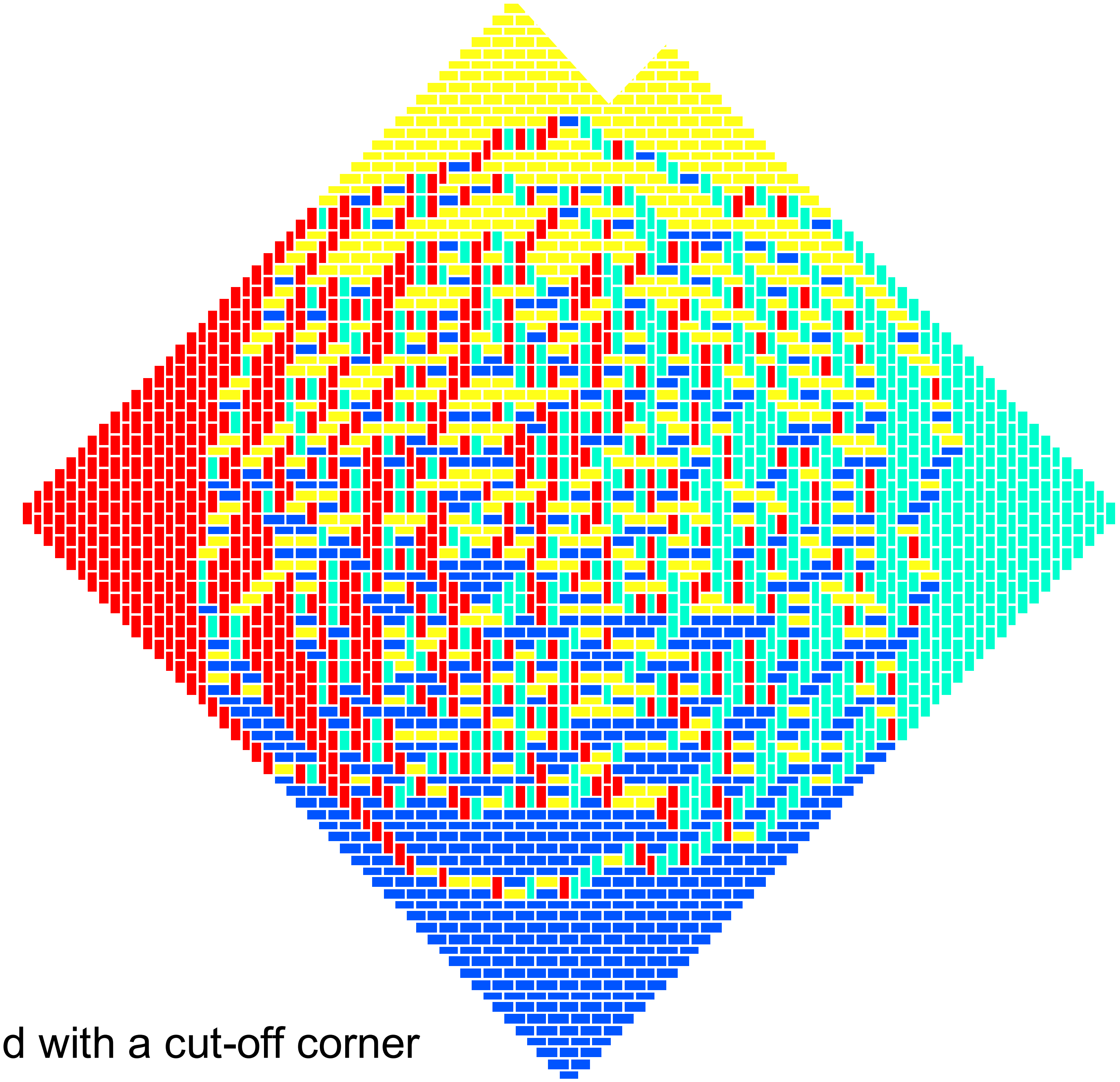
Convex region	vs	Concave region
Smooth curve	vs	Singular curve



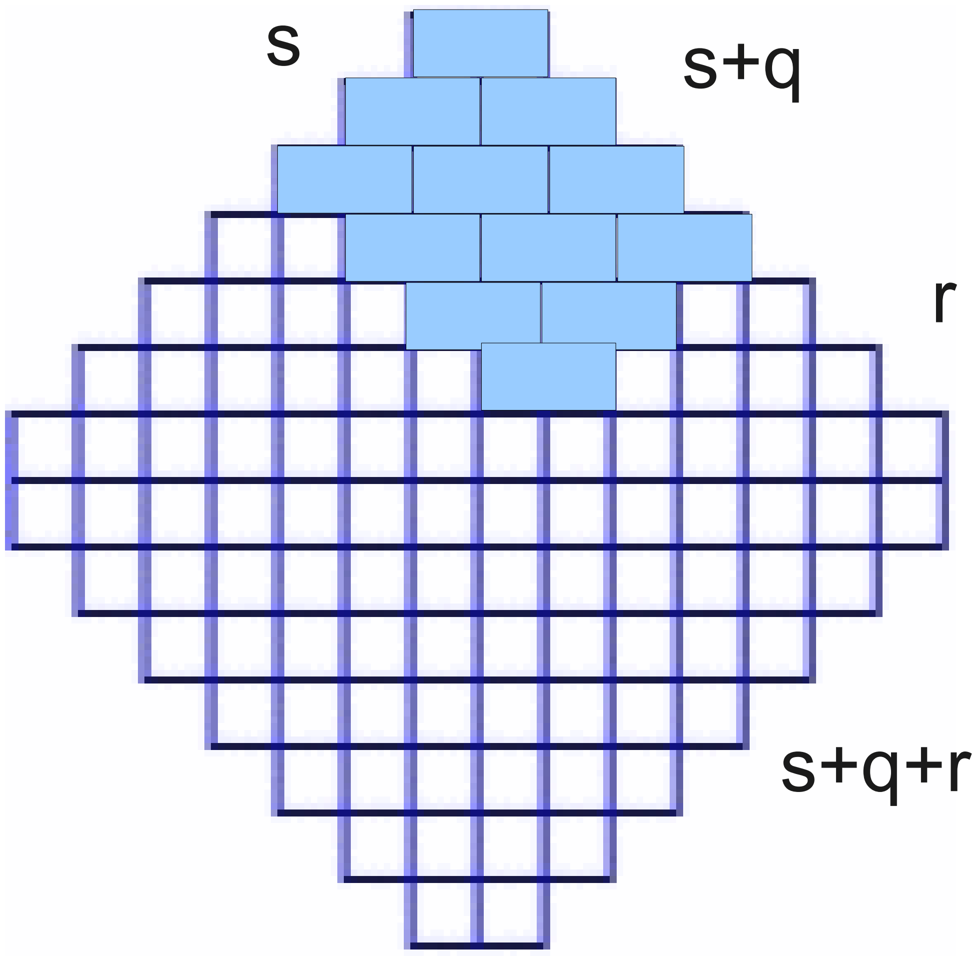
Rhombi tilings of an hexagon with an erased corner [Kenyon, Okounkov '05]



Rhombi tilings of an hexagon with an erased corner [Kenyon, Okounkov '05]



Aztec Diamond with a cut-off corner



$s = 3$

$q = 1$

$r = 3$

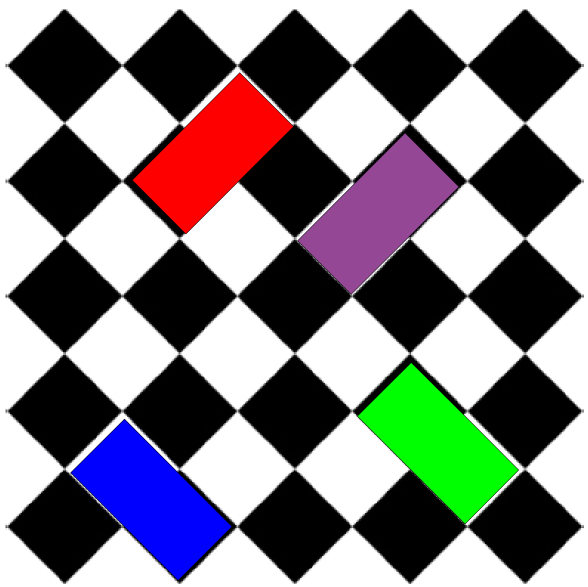
$s + r + q = 7$

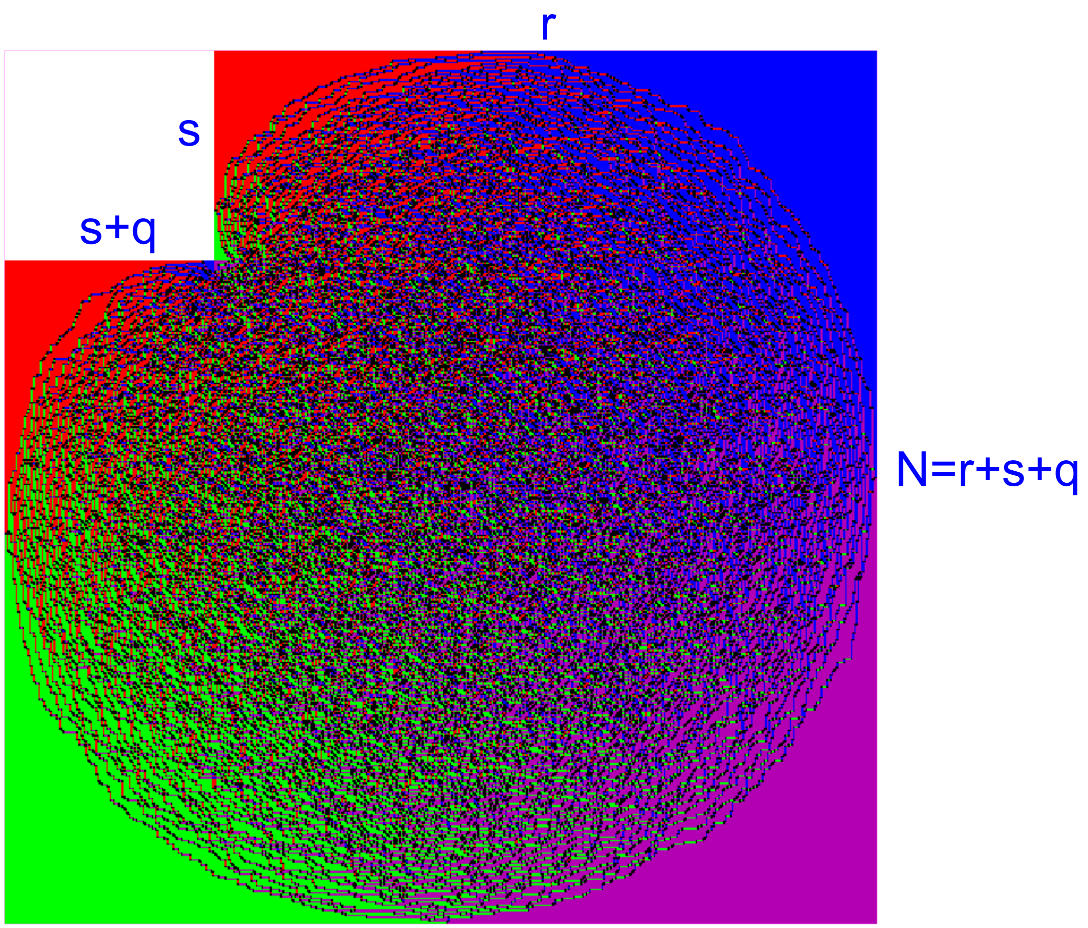
Some numerical results

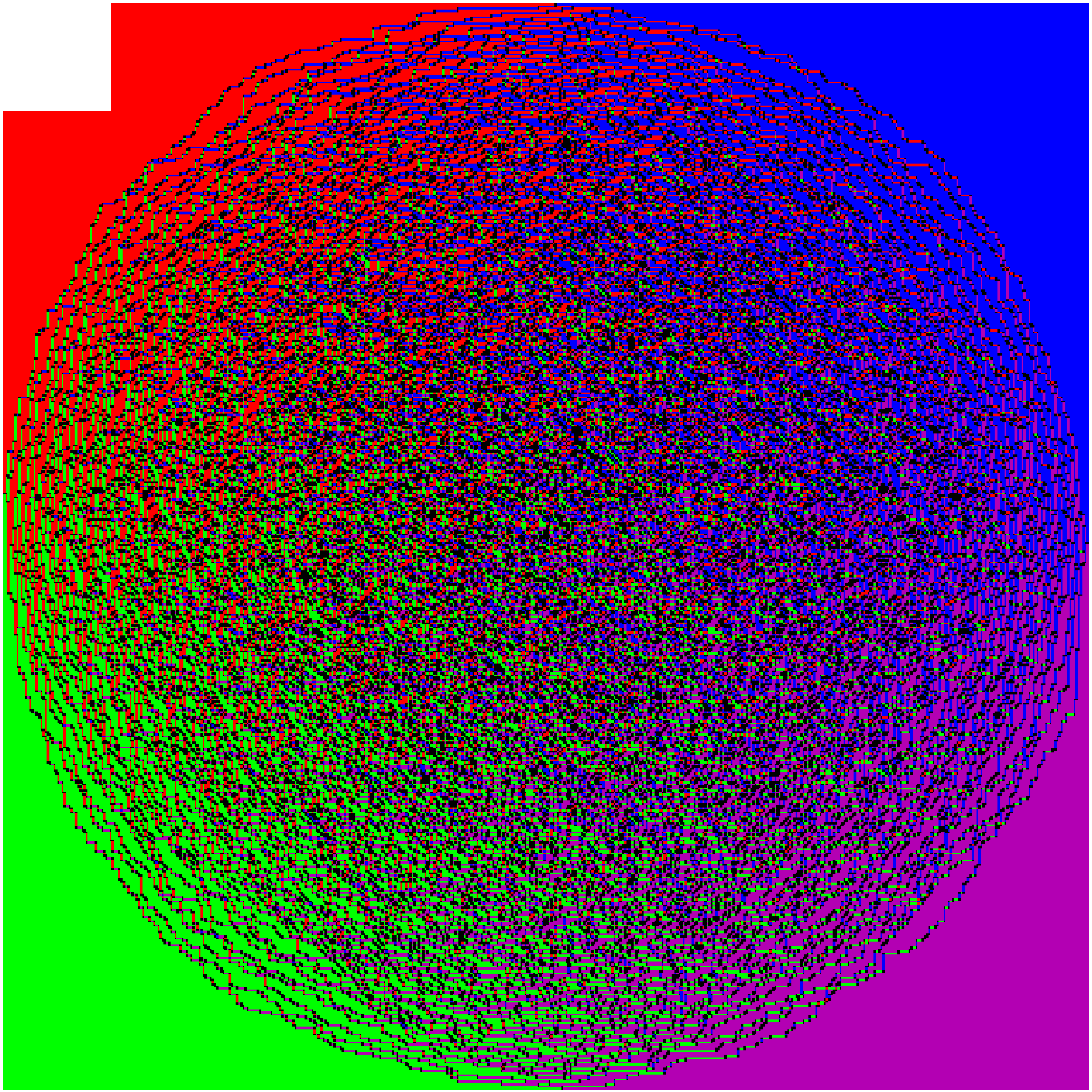
[FC-Sportiello, to appear]

- Pictures are produced with a C code based on a version kindly provided by Ben Wieland, exploiting the `Coupling From The Past' algorithm [Propp-Wilson '96].

- We freeze a rectangular region of size $(s+q) \times s$ in the top-left corner (we restrict to the symmetric situation $q = 0$, for simplicity)





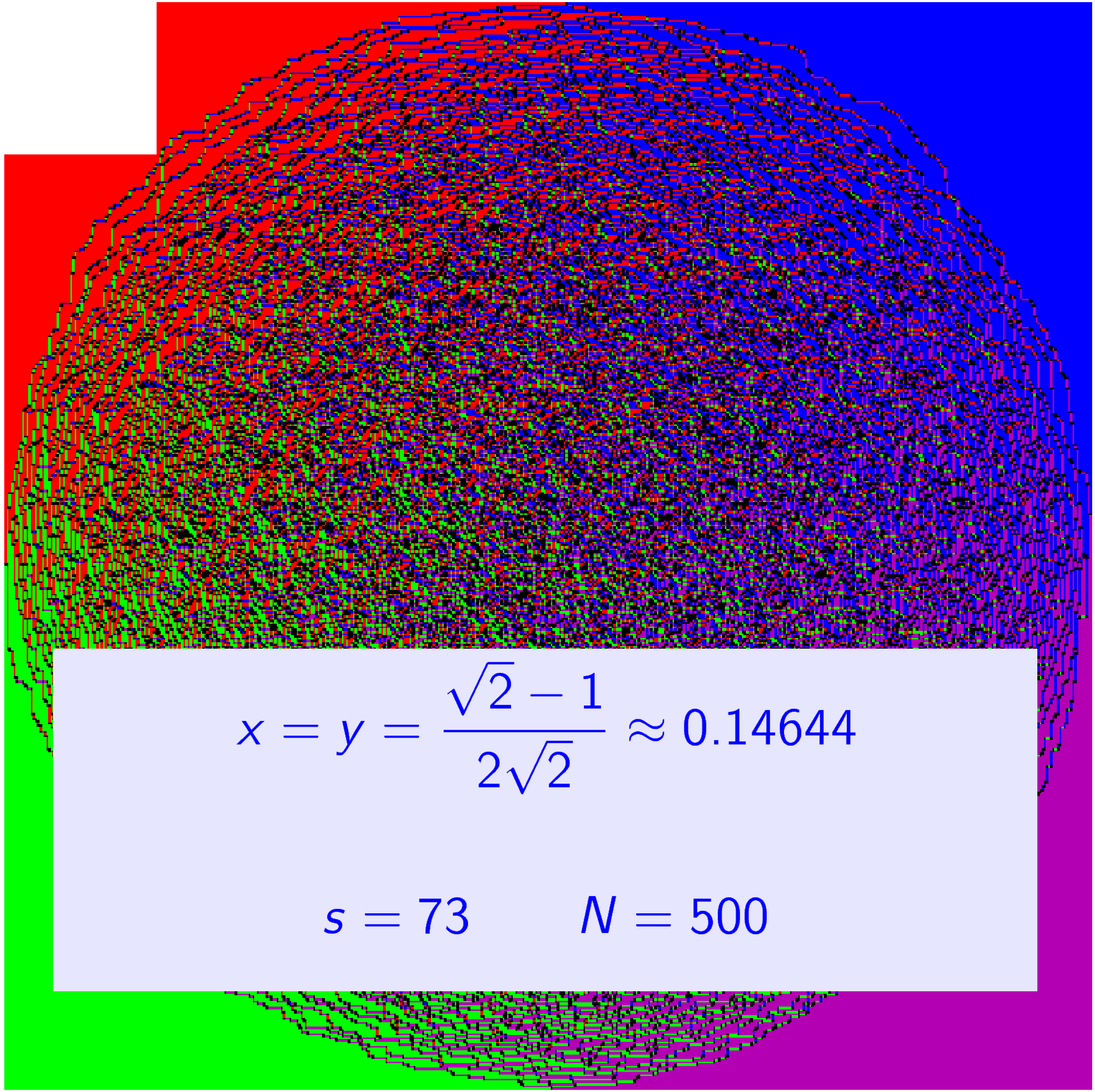


$N = 500$
$s = 50$

$q = 0$
$r = N - s$

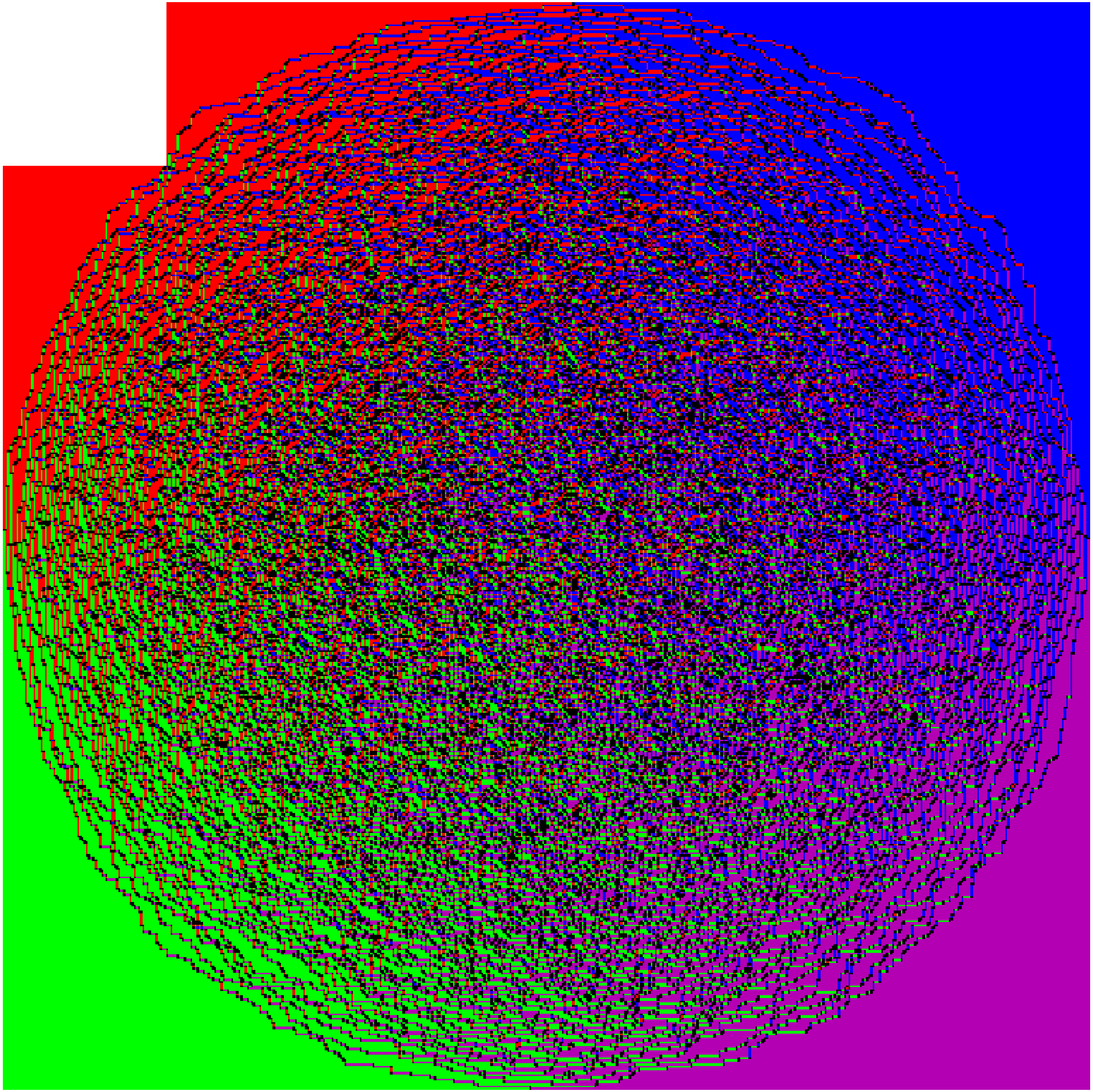
$N = 500$
$s = 70$

$q = 0$
$r = N - s$



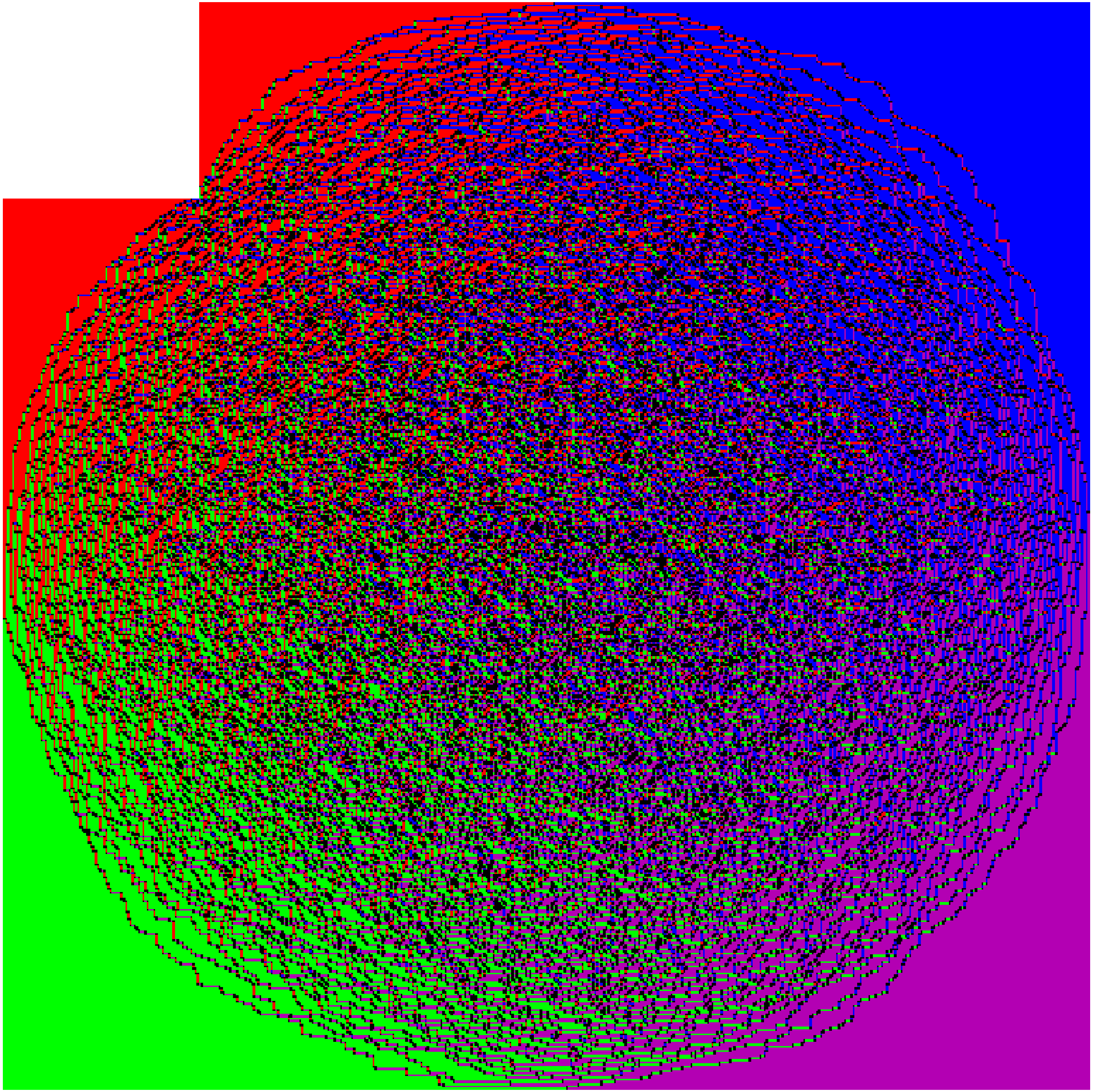
$$x = y = \frac{\sqrt{2} - 1}{2\sqrt{2}} \approx 0.14644$$

$s = 73 \qquad N = 500$



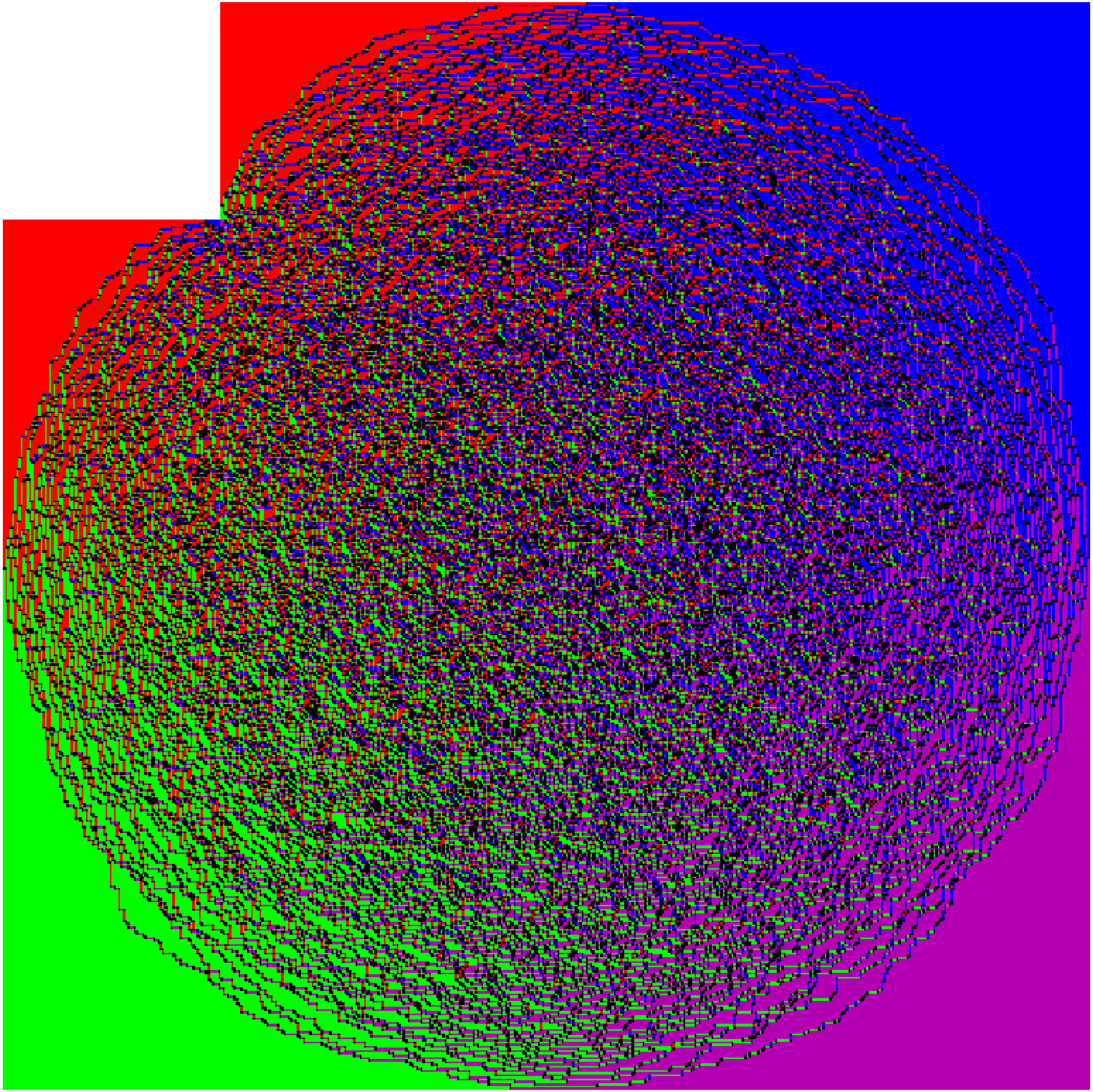
$N = 500$
$s = 75$

$q = 0$
$r = N - s$



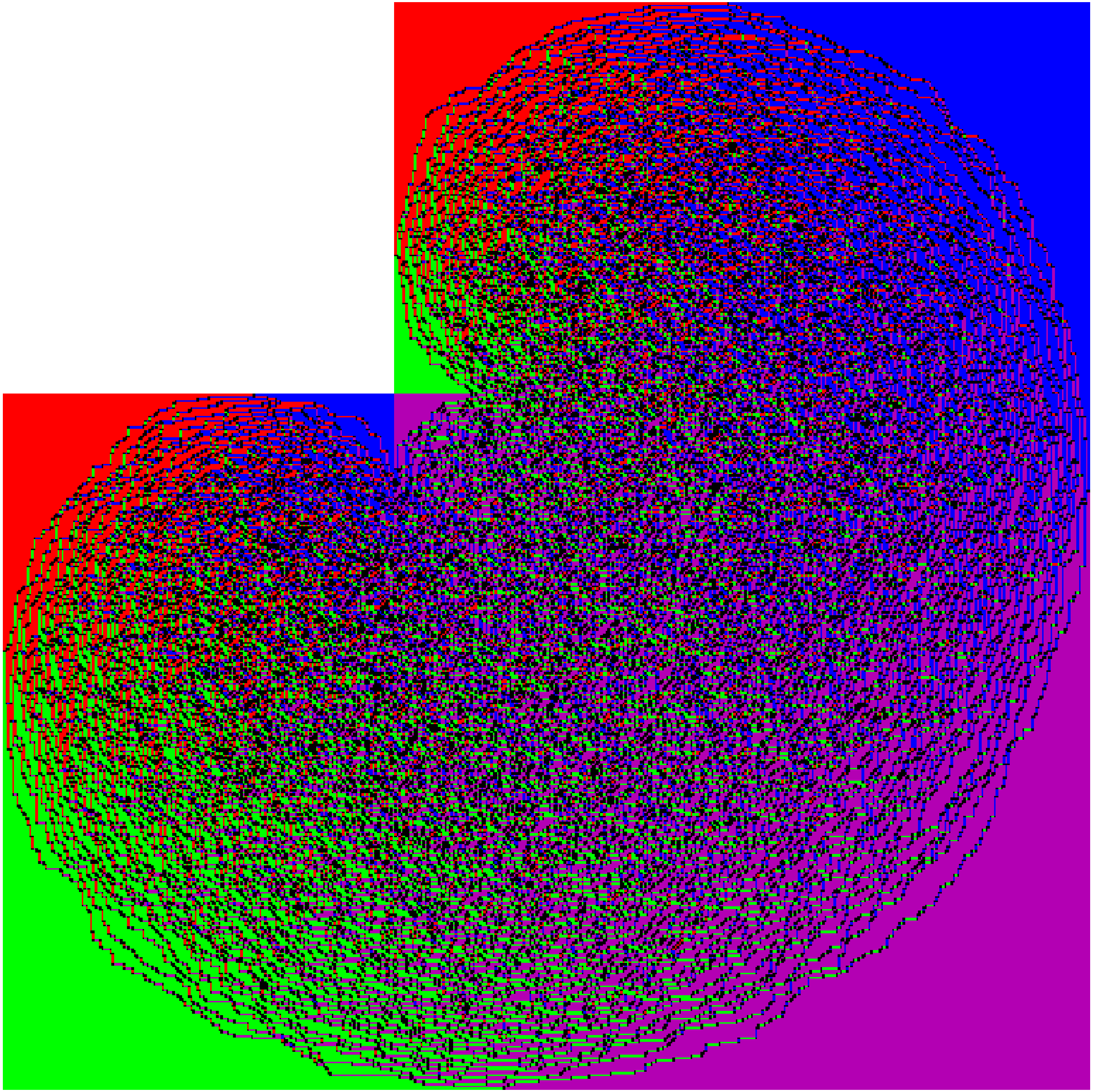
$N = 500$
$s = 90$

$q = 0$
$r = N - s$



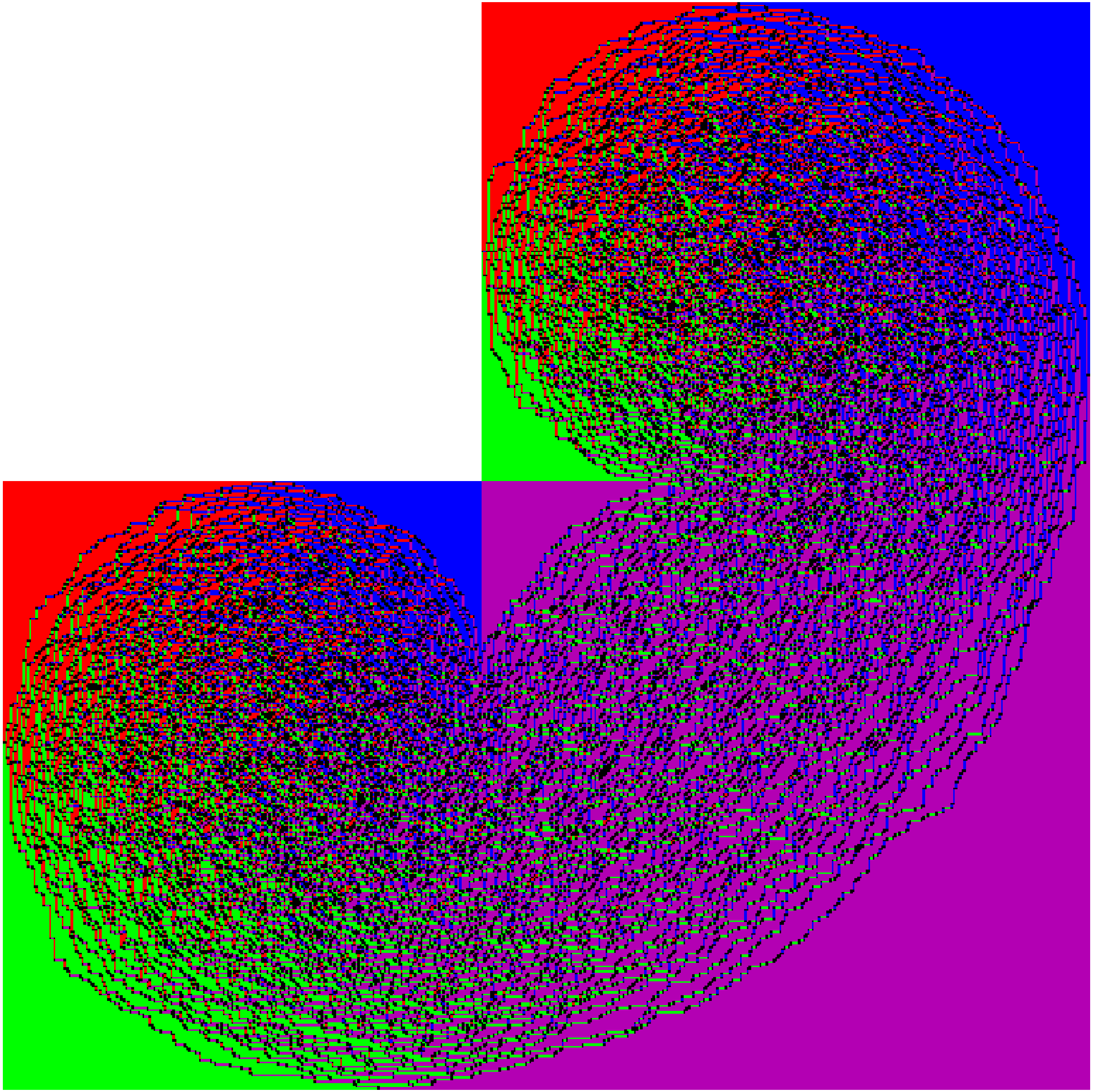
$N = 500$
$s = 100$

$q = 0$
$r = N - s$



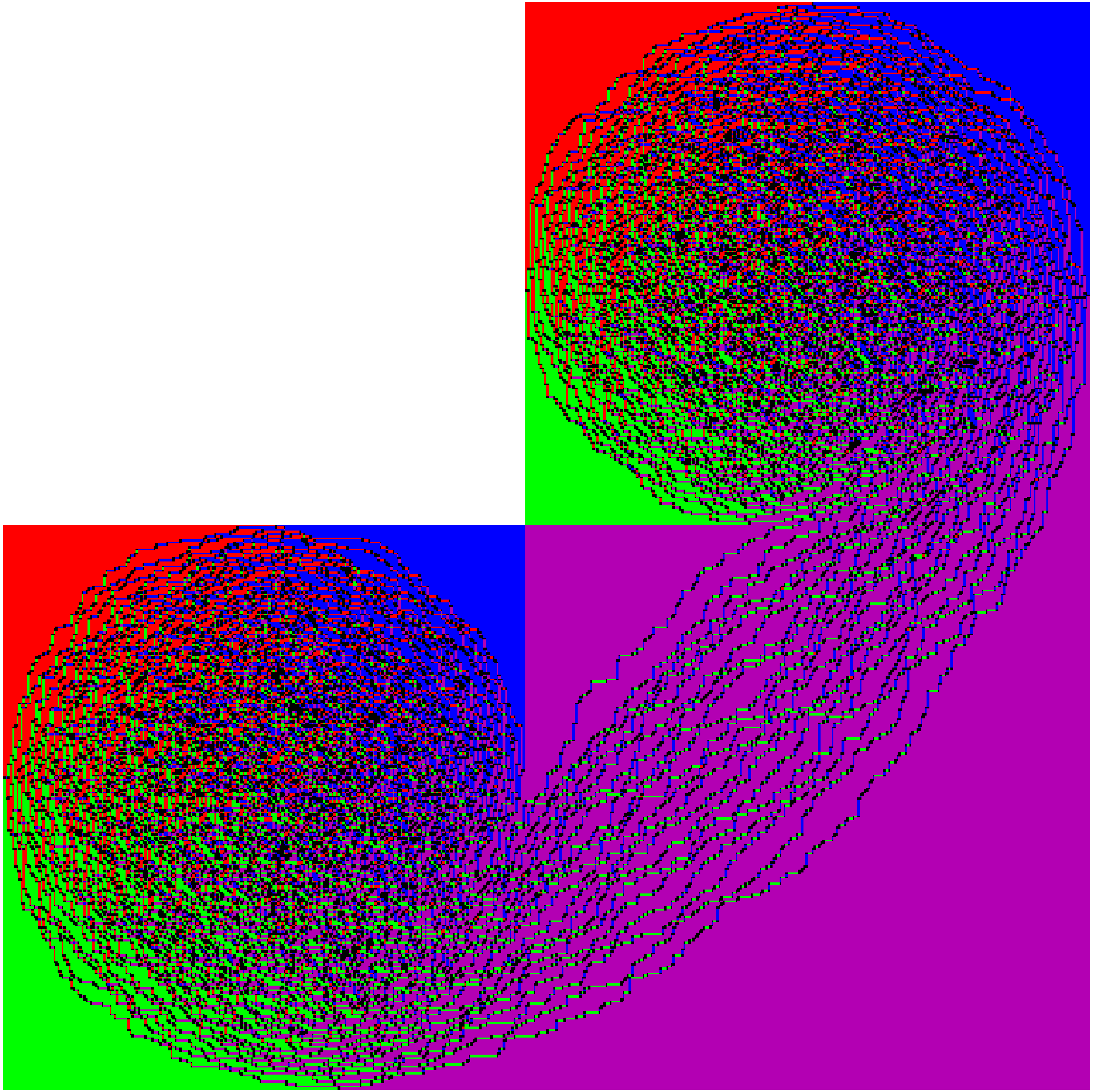
$N = 500$
$s = 180$

$q = 0$
$r = N - s$



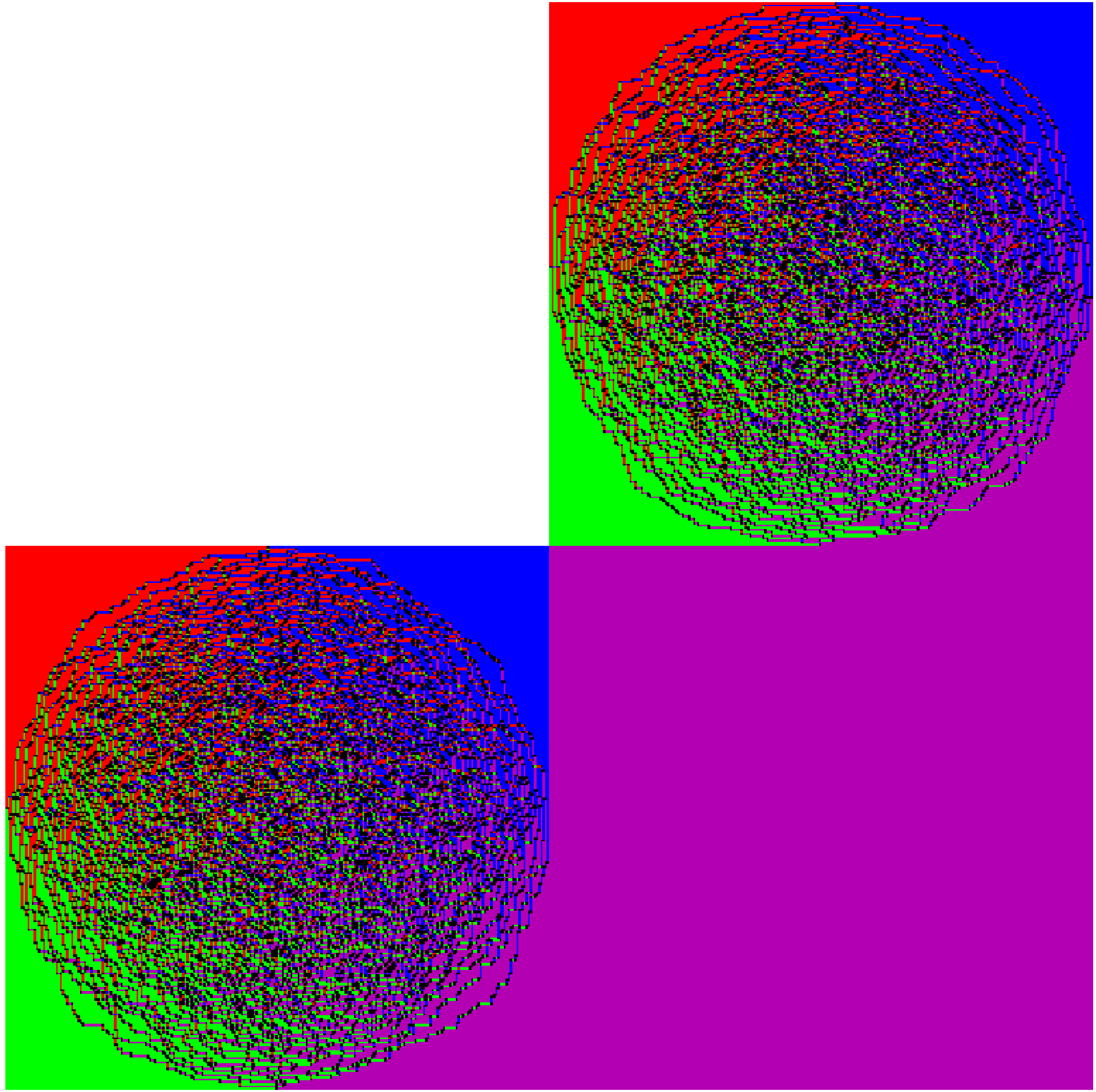
$N = 500$
$s = 220$

$q = 0$
$r = N - s$



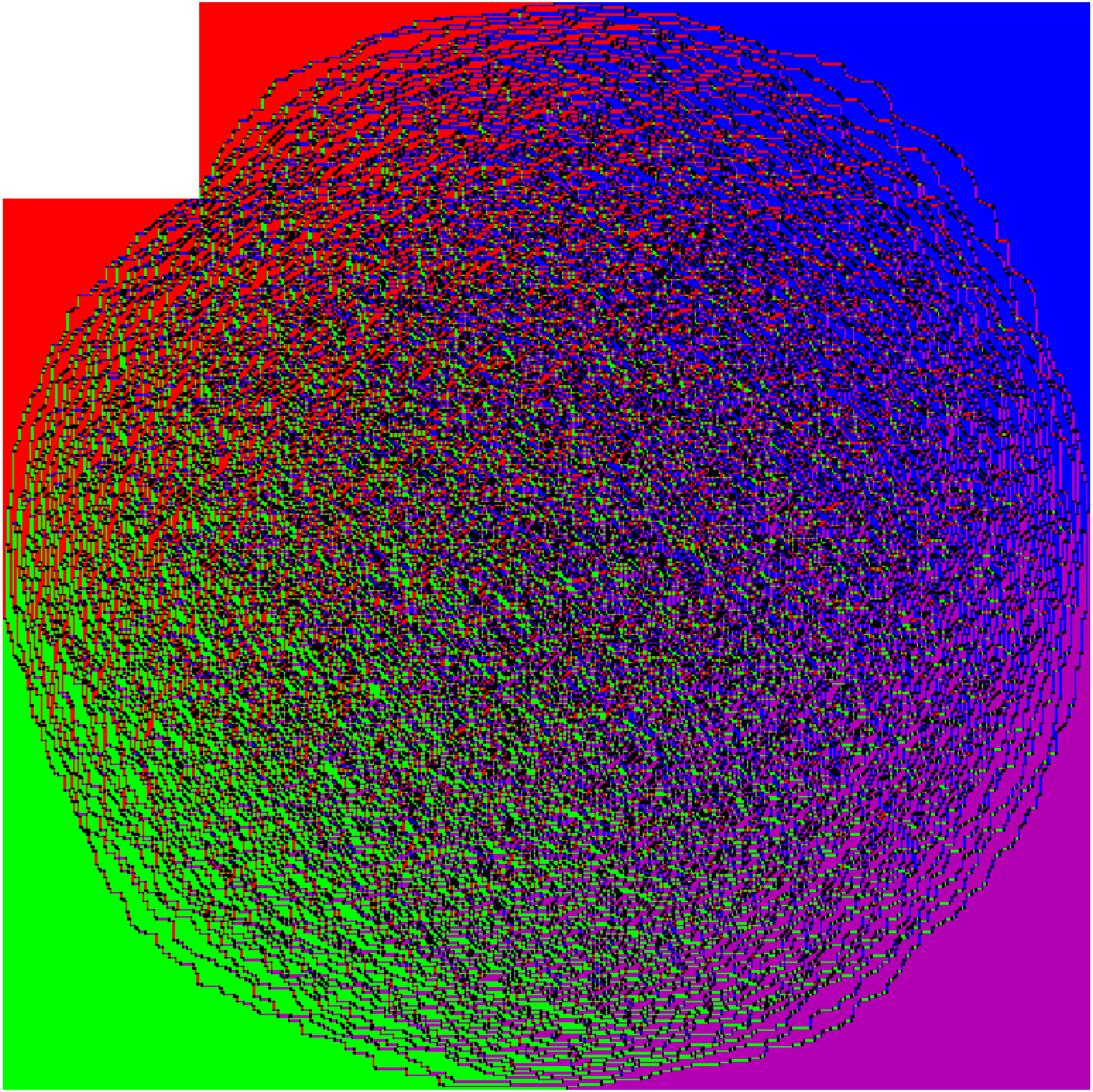
$N = 500$
$s = 240$

$q = 0$
$r = N - s$



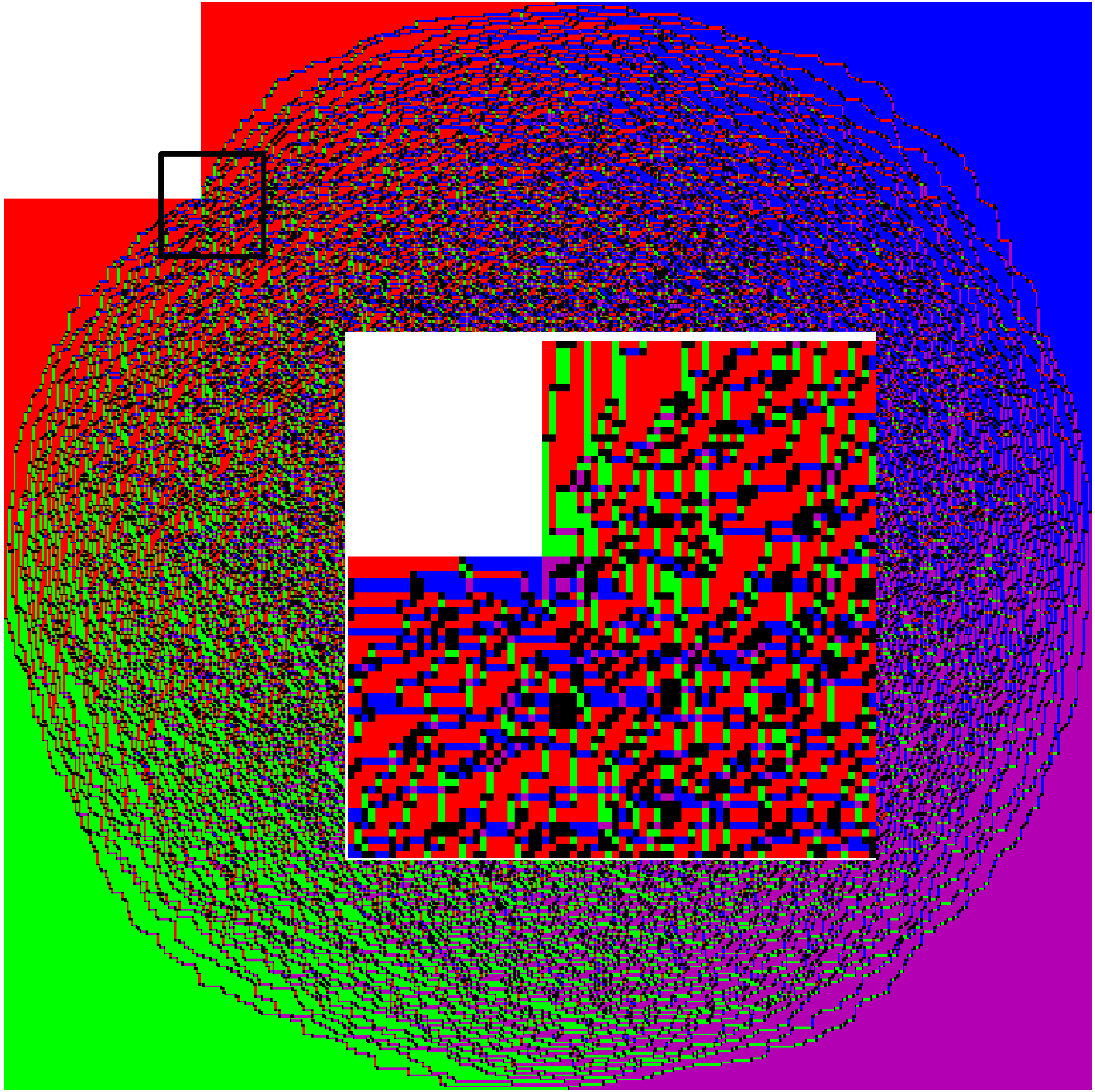
$N = 500$
$s = 250$

$q = 0$
$r = N - s$



$N = 500$
$s = 90$

$q = 0$
$r = N - s$



$N = 500$
$s = 90$

$q = 0$
$r = N - s$

Define:

$$Z_{r,s,q} \quad := \quad \text{number of tilings of } AD_{r,s,q}$$

1) Compute $Z_{r,s,q}$ for arbitrary r, s, q integers

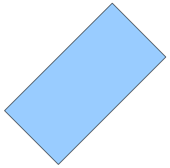
2) Investigate the behaviour of $Z_{r,s,q}$ in the scaling limit:

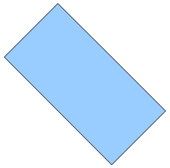
$$r, s, q \to \infty \qquad \text{with} \qquad v := \tfrac{s}{r} \qquad \text{and} \qquad u := \tfrac{q}{s} \qquad \text{fixed}$$

In particular, evaluate the `free energy per domino':

$$F(v, u) := - \lim_{\substack{r,s,q \to \infty \\ s/r = v,\, q/s = u}} \frac{\log Z_{r,s,q}}{(r+s+q)^2 - s(s+q)}$$

Introduce bias (weighted enumeration)

To each NE-SW domino  assign weight: $\sqrt{2(1-\alpha)}$

To each NW-SE domino  assign weight: $\sqrt{2\alpha}$

$\alpha \in [0,1]$

($\alpha = \frac{1}{2}$ corresponds to the uniform measure)

For generic α , the Arctic Circle becomes an Arctic Ellipse.

Define:

$Z_{r,s,q}(\alpha)$:= weighted number of tilings of $AD_{r,s,q}$

1) Compute $Z_{r,s,q}(\alpha)$ for arbitrary r, s, q integers

2) Investigate the behaviour of $Z_{r,s,q}(\alpha)$ in the scaling limit:

$r, s, q \to \infty$ with $v := \frac{s}{r}$ and $u := \frac{q}{s}$ fixed

In particular, evaluate the `free energy per domino':

$$F(v, u) := -\lim_{\substack{r,s,q\to\infty \\ s/r=v,q/s=u}} \frac{\log Z_{r,s,q}(\alpha)}{(r+s+q)^2 - s(s+q)}$$

Recall:

$Z_{r,s,q}(\alpha)$:= weighted number of tilings of $AD_{r,s,q}$

$Z_n(\alpha)$:= weighted number of tilings of AD_n

= $2^{n(n+1)/2}$ (NB: does not depend on α)

Define: $$f_{r,s,q} := \frac{Z_{r,s,q}(\alpha)}{Z_{r+s+q}(\alpha)}[2(1-\alpha)]^{s(s+q)/2}$$

Theorem:

[FC-Pronko'08] [Pronko'13]

$$f_{r,s,q} = \prod_{k=0}^{s-1} \frac{q!}{k!(q+k)!} \cdot \frac{(1-\alpha)^{s(s+q)}}{\alpha^{s(s-1)/2}} \cdot \det_{1\le j,k\le s} \left[\sum_{m=0}^{r-1} m^{j+k-2} \frac{(q+m)!}{q!m!} \alpha^m \right]$$

- Proof relies upon the correspondence of domino tilings of Aztec Diamond with the six-vertex model with Domain Wall boundary conditions.
- Main ingredient in the derivation is the Quantum Inverse Scattering Method.
- $f_{r,s,q}$ is the probability of observing, in a random tiling of the plain Aztec Diamond, a `frozen' corner region of size $s \times (s+1)$
- Actually this is a particular case of a more general formula holding for ASM, six-vertex model, etc...

Some properties of $f_{r,s,q}$

[Pronko'13]

$$f_{r,s,q} = \prod_{k=0}^{s-1} \frac{q!}{k!(q+k)!} \cdot \frac{(1-\alpha)^{s(s+q)}}{\alpha^{s(s-1)/2}} \cdot \det_{1\le j,k\le s} \left[\sum_{m=0}^{r-1} m^{j+k-2} \frac{(q+m)!}{q!m!} \alpha^m \right]$$

(Hankel Determinant)

Introducing *'time'* via $\alpha = e^{-t}$, and restricting to the determinant:

$$\tau_s(r,q,\alpha) := \det_{1\le j,k\le s} \left[\sum_{m=0}^{r-1} m^{j+k-2} \frac{(q+m)!}{q!m!} \alpha^m \right] = \det_{1\le j,k\le s} \left[\left(\frac{\partial}{\partial t} \right)^{j+k-2} \phi(t) \right]$$

Sylvester identity for determinants immediately implies:

$$\partial_t^2 \log \tau_s = \frac{\tau_{s+1}\tau_{s-1}}{\tau_s^2}$$

$\tau_s := \tau_s(r,q,\alpha)$ is the tau-function of a semi-infinite Toda chain.

Some properties of $f_{r,s,q}$

$$f_{r,s,q} = \prod_{k=0}^{s-1} \frac{q!}{k!(q+k)!} \cdot \frac{(1-\alpha)^{s(s+q)}}{\alpha^{s(s-1)/2}} \cdot \det_{1\le j,k\le s} \left[\sum_{m=0}^{r-1} m^{j+k-2} \frac{(q+m)!}{q!m!} \alpha^m \right]$$

(Hankel Determinant)

$$\mathcal{D}_\alpha^2 \log f_{r,s,q} = \frac{r(r+q)\alpha}{(1-\alpha)^2} \left(\frac{f_{r+1,s,q} f_{r-1,s,q}}{f_{r,s,q}^2} - 1 \right)$$

$$\mathcal{D}_\alpha \log \frac{f_{r,s,q+1}}{f_{r,s,q}} = \frac{r\alpha}{1-\alpha} \left(\frac{f_{r+1,s,q} f_{r-1,s,q+1}}{f_{r,s,q} f_{r,s,q+1}} - 1 \right)$$

(s fixed)

$$\mathcal{D}_\alpha \log \frac{f_{r+1,s,q}}{f_{r,s,q+1}} = \frac{r+q+1}{1-\alpha} \left(\frac{f_{r+1,s,q+1} f_{r,s,q}}{f_{r,s,q+1} f_{r+1,s,q}} - 1 \right)$$

$$\mathcal{D}_\alpha := \alpha \frac{\partial}{\partial \alpha}$$

$$\mathcal{D}_\alpha^2 \log f_{r,s,q} = \frac{s(s+q)\alpha}{(1-\alpha)^2} \left(\frac{f_{r,s+1,q} f_{r,s-1,q}}{f_{r,s,q}^2} - 1 \right)$$

$$\mathcal{D}_\alpha \log \frac{f_{r,s,q+1}}{f_{r,s,q}} = \frac{s\alpha}{1-\alpha} \left(\frac{f_{r,s+1,q} f_{r,s-1,q+1}}{f_{r,s,q} f_{r,s,q+1}} - 1 \right)$$

(r fixed)

$$\mathcal{D}_\alpha \log \frac{f_{r,s+1,q}}{f_{r,s,q+1}} = \frac{s+q+1}{1-\alpha} \left(\frac{f_{r,s+1,q+1} f_{r,s,q}}{f_{r,s,q+1} f_{r,s+1,q}} - 1 \right)$$

(Toda-like differential equations)

Some properties of $f_{r,s,q}$

$$f_{r,s,q} = \prod_{k=0}^{s-1} \frac{q!}{k!(q+k)!} \cdot \frac{(1-\alpha)^{s(s+q)}}{\alpha^{s(s-1)/2}} \cdot \det_{1\le j,k\le s} \left[\sum_{m=0}^{r-1} m^{j+k-2} \frac{(q+m)!}{q!m!} \alpha^m \right]$$

(Hankel Determinant)

Following in spirit [Zinn-Justin'00], one can rewrite:

$$\tau_s(r,q,\alpha) := \det_{1\le j,k\le s} \left[\sum_{m=0}^{r-1} m^{j+k-2} \frac{(q+m)!}{q!m!} \alpha^m \right]$$

$$= \frac{1}{s!} \sum_{m_1=0}^{r-1} \cdots \sum_{m_s=0}^{r-1} \prod_{1\le j<k\le s} (m_k - m_j)^2 \prod_{j=1}^{s} \binom{q+m_j}{q} \alpha^{m_j}$$

(Random Matrix Model with discrete measure)

$\tau_s(r, q, \alpha)$ at some special values of parameters

$$\tau_s(r,q,\alpha) := \det_{1\le j,k\le s}\left[\sum_{m=0}^{r-1} m^{j+k-2}\frac{(q+m)!}{q!m!}\alpha^m\right]$$

- $r \to \infty$

$$\lim_{r\to+\infty}\tau_s(r,q,\alpha) = \prod_{j=0}^{s-1}\frac{(q+j)!j!}{q!}\cdot\frac{\alpha^{s(s-1)/2}}{(1-\alpha)^{s(s+q)}} \qquad \text{(Meixner)}$$

•

- $r = s$

$$\tau_r(r,q,\alpha) = \prod_{k=0}^{s-1}\frac{k!(k+q)!}{q!}\cdot\alpha^{s(s-1)/2} \qquad \text{(chose } m_j = j\text{)}$$

- $\alpha \to 0$

$$\tau_s(r,q,\alpha) \sim \prod_{k=0}^{s-1}\frac{k!(k+q)!}{q!}\cdot\alpha^{s(s-1)/2} \qquad \text{as } \alpha\to 0$$

- $\alpha = 1$

$$\tau_s(r,q,1) = \prod_{j=0}^{s-1}\frac{(j!(j+q)!)^2(j+q+r)!}{q!(r-j-1)!(2j+q)!(2j+q+1)!} \qquad \text{(Hahn)}$$

Free energy: differential eq. approach

For simplicity we restrict to the symmetric situation $q = 0$.

We define $f_{r,s} := f_{r,s,q}\big|_{q=0}$, satisfying:

$$\mathcal{D}_\alpha^2 \log f_{r,s} = \frac{r^2\alpha}{(1-\alpha)^2}\left(\frac{f_{r+1,s}f_{r-1,s}}{f_{r,s}^2} - 1\right)$$

$$\mathcal{D}_\alpha^2 \log f_{r,s} = \frac{s^2\alpha}{(1-\alpha)^2}\left(\frac{f_{r,s+1}f_{r,s-1}}{f_{r,s}^2} - 1\right)$$

Following [Korepin-Zinn-Justin'00], we assume

$$\log f_{r,s} = -r^2\sigma(v;\alpha) + o(r^2) \qquad r,s\to\infty \qquad \frac{s}{r} = v\,, \qquad v\in[0,1]$$

We get for $\sigma(v;\alpha)$:

$$\mathcal{D}_\alpha^2 \sigma = -\frac{\alpha}{(1-\alpha)^2}\left(\mathrm{e}^{-2\sigma+2v\sigma'-v^2\sigma''} - 1\right)$$

$$\mathcal{D}_\alpha^2 \sigma = -\frac{v^2\alpha}{(1-\alpha)^2}\left(\mathrm{e}^{-\sigma''} - 1\right)$$

We want to solve for $\sigma(v;\alpha)$ in the domain $v\in[0,1]$, $\alpha\in[0,1]$.

NB: free energy density is given by: $\dfrac{\sigma(v;\alpha)+\frac{1}{2}v^2\log(1-\alpha)}{1+2v}$

Initial and boundary conditions

- $v = 0$

$$\sigma(0; \alpha) = 0$$

- $v = 1$

$$\sigma(1; \alpha) = -\log(1 - \alpha)$$

- $\alpha \to 0$

$$\sigma(v; 0) = 0$$

- $\alpha \to 1$

$$\lim_{\alpha \to 1^-} \left[\sigma(v; \alpha) + v^2 \log\left(\frac{1 - \alpha}{\sqrt{\alpha}} \right) \right] = -v^2 \tilde{\sigma}(v)$$

where

$$\tilde{\sigma}(v) := \lim_{s \to \infty} \frac{1}{s^2} \log \frac{\tau_s(\frac{s}{v}, 1)}{\prod_{j=0}^{s-1} (j!)^2}$$

is given by:

$$\tilde{\sigma}(v) = \frac{1}{2} \left[\log 16 v^2 - \frac{(1 - v)^2}{v^2} \log(1 - v) - \frac{(1 + v)^2}{v^2} \log(1 + v) \right]$$

Solution of the differential equations ($q = 0$)

The derivation is technically involved and not particularly interesting.

The final solution is almost trivial, the non trivial contribution coming from the integration constant $\tilde{\sigma}(v)$

We have the following result:

$$\sigma(v, \alpha) = 0 \qquad\qquad v \in [0, v_c(\alpha)]$$

$$\sigma(v; \alpha) = \frac{1}{2}\left[v^2 \log\frac{v^2}{v_c^2} - (1-v)^2 \log\frac{1-v}{1-v_c} - (1+v)^2 \log\frac{1+v}{1+v_c}\right]$$

$$v \in [v_c(\alpha), 1]$$

where

$$v_c = v_c(\alpha) = \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}$$

Free energy: Random Matrix Model approach

Again we restrict for simplicity to the symmetric situation $q = 0$.
Following [Zinn-Justin'00], we define:

$$\tau_s(r,\alpha) := \tau_s(r,0,\alpha) \quad = \quad \frac{1}{s!}\sum_{m_1=0}^{r-1}\cdots\sum_{m_s=0}^{r-1}\prod_{1\le j<k\le s}(m_k-m_j)^2\prod_{j=1}^{s}\alpha^{m_j}$$

Hermitean $s \times s$ random matrix integral (with a discrete measure)
[Douglas-Kazakov'93]

To investigate the large s behaviour of $\tau_s(r,\alpha)$, one need to rescale:

$$m_k \to \mu_k := \frac{m_k}{s}, \qquad k = 1,\ldots,s$$

In the large s limit, sums can now be reinterpreted as Riemann sums, and replaced by integrals:

$$\tau_s(r,\alpha) \propto \int_0^c d^s\mu \prod_{1\le j<k\le s}(\mu_k-\mu_j)^2\prod_{j=1}^{s}\alpha^{s\mu_j}$$

where $c := r/s = 1/v$

Saddle-point approximation

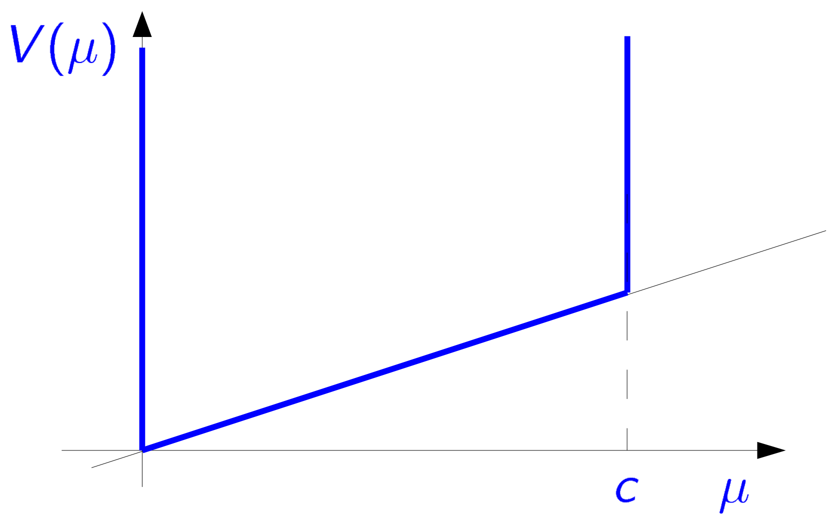
Write the integrand as:

$$\exp\left[\sum_{\substack{i,j=1\\ i\neq j}}^{s} \log|\mu_j-\mu_k| + s\log\alpha\sum_{j=1}^{s}\mu_j\right]$$

Saddle-point eqs. read:

$$2\sum_{\substack{k=1\\ k\neq j}}^{s}\frac{1}{\mu_j-\mu_k} = -\log\alpha \qquad\qquad j=1,\ldots,s$$

The solution of the saddle-point eqs. is given by the equilibrium configuration of a set of mutually repelling charged particles, in the linear potential $V(\mu)=-\mu\log\alpha$, confined to the real interval $\mu\in[0,c]$:



Saddle-point approximation

Introduce a normalized density of solutions of saddle-point eqs.:

$$\mu_k^* \to \mu(x) := \mu\left(\frac{k}{s}\right), \qquad \rho(\mu) = \frac{1}{d\mu(x)/dx}, \qquad \int_0^c \rho(\mu)d\mu = 1$$

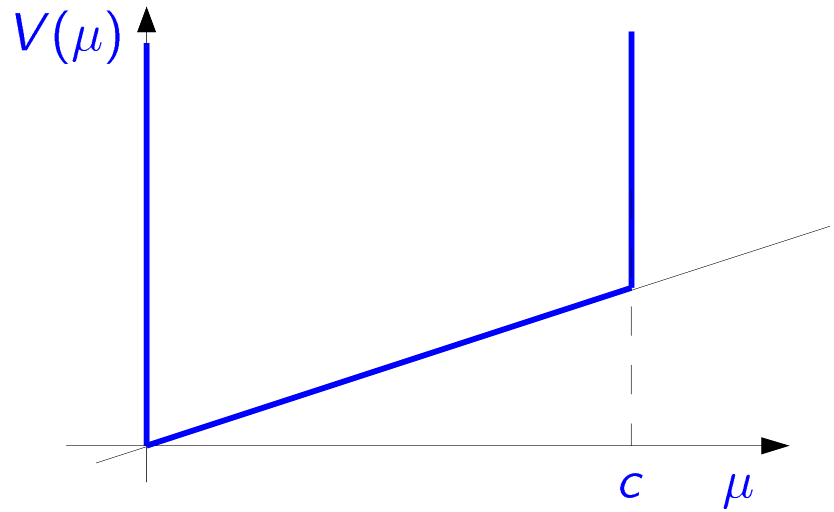
Discreteness of eigenvalues implies $\rho(\mu) \leq 1$

Standard methods (e.g. using the resolvent) can be exploited to determine $\rho(\mu)$ and solve the model.

The only caveat is the implementation of the constraint: $\rho(\mu) \leq 1$

(in fact, just a minor technical complication)

[Douglas-Kazakov'93],[Brézin-Kazakov'99],[Zinn-Justin'00]



NB: We have two `hard walls' at $\mu = 0$ and $\mu = c$

Near the edges of its support the density as a universal behaviour:

If $Supp(\rho) = [a, b]$, then, e.g. in the vicinity of a :

$V'(a)$ finite $\qquad \rho(z) \sim \sqrt{z-a}$

'hard wall'at $z = a$ $\qquad \rho(z) \sim \dfrac{1}{\sqrt{z-a}}$

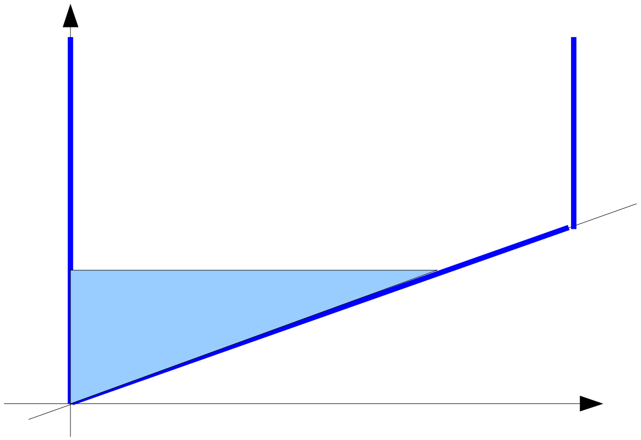
The discreteness constraint $\rho(z) \leq 1$ thus implies:

In the vicinity of an hard wall: $\rho(\mu) = 0$ or $\rho(\mu) = 1$

Two scenarios

- *i*) large c (or small α): potential well is deep and narrow

The eigenvalues accumulate to the left:

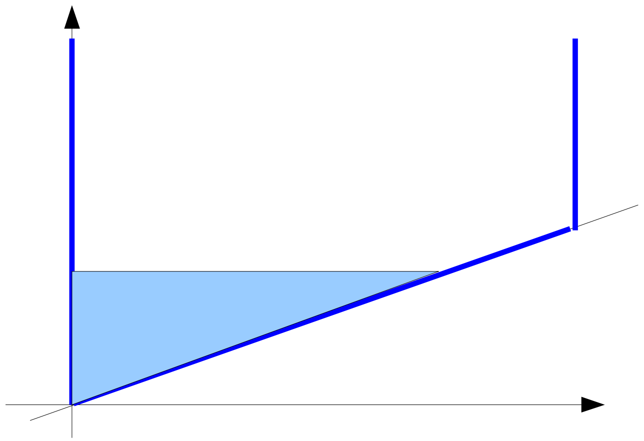


$\rho(0) = 1 \qquad \rho(c) = 0$

Two scenarios

- *i*) large c (or small α): potential well is deep and narrow

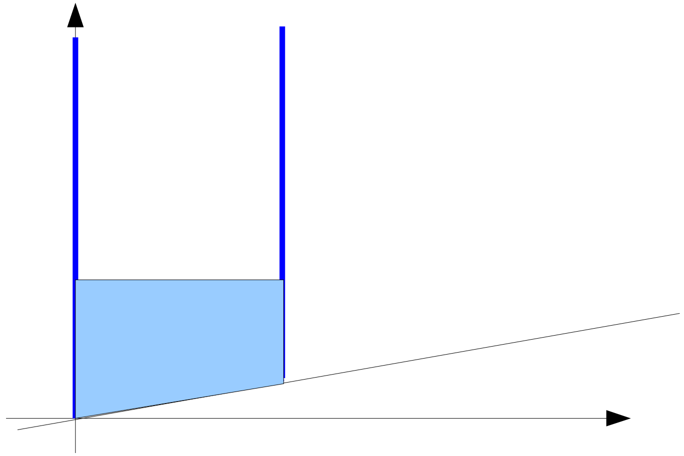
The eigenvalues accumulate to the left:



$$\rho(0) = 1 \qquad \rho(c) = 0$$

- *ii*) small c (or large α): potential well is wide and not so deep

The eigenvalues expand till the right wall:

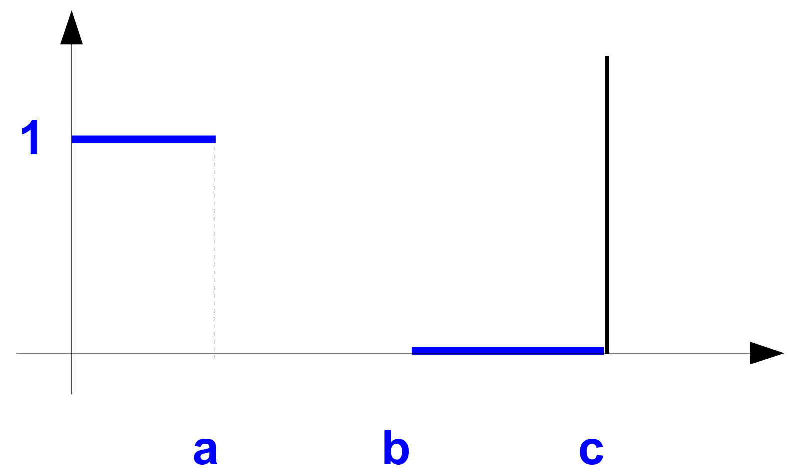


$$\rho(0) = 1 \qquad \rho(c) = 1$$

Scenario i)

We have: $\rho(0) = 1$ and $\rho(c) = 0$, thus:

$$\begin{array}{ll} \rho(\mu) = 1 & \mu \in [0, a] \\ 0 < \rho(\mu) < 1, & \mu \in [a, b] \\ \rho(\mu) = 0, & \mu \in [b, c] \end{array}$$



Solving the saddle-point eqs determines endpoints a and b and density $\rho(z)$:

$$a = \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}, \qquad b = \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}} \qquad \rho(z) = \frac{2}{\pi} \arctan \frac{\sqrt{a(b-z)}}{\sqrt{b(z-a)}} \qquad z \in [a, b]$$

NB1: Exactly the same Random Matrix Model appears: (but with $c = \infty$):

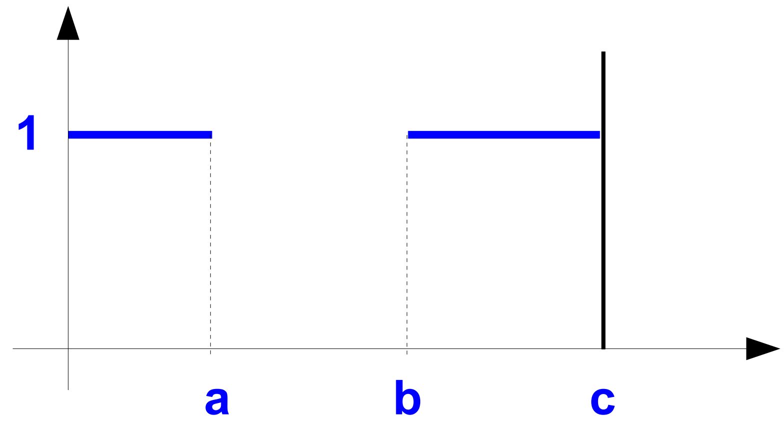
- in [Brezin-Kazakov'00] (statistics of partitions for the permutation group)
- in [Zinn-Justin'00], (ferroelectric phase ($|\Delta| > 1$) of the DW 6VM partition function.

NB2: This scenario holds as long as $c > b$ $\Rightarrow$ $c_{crit} = \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}$

Scenario ii)

We have: $\rho(0) = 1$ and $\rho(c) = 1$, thus:

$$\rho(z) = 1 \qquad z \in [0, a]$$
$$0 < \rho(z) < 1, \qquad z \in [a, b]$$
$$\rho(z) = 1, \qquad z \in [b, c]$$

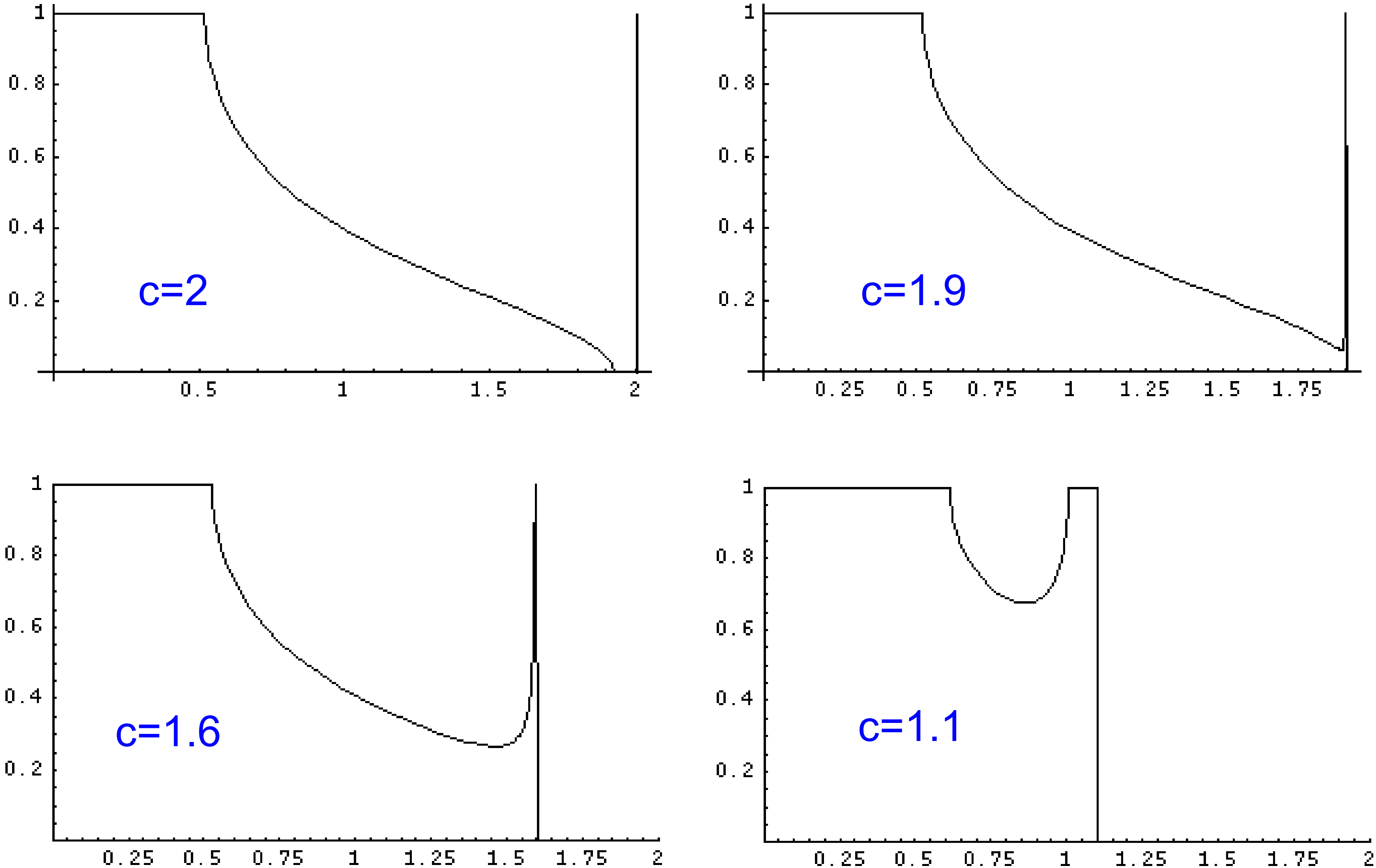


Again, solving saddle-point eqs determines endpoints a and b and density $\rho(z)$:

$$a = \frac{[\sqrt{c+1} - \sqrt{(c-1)\sqrt{\alpha}}]^2}{2(1+\sqrt{\alpha})}, \qquad b = \frac{[\sqrt{c+1} + \sqrt{(c-1)\sqrt{\alpha}}]^2}{2(1+\sqrt{\alpha})}$$

$$\rho(z) = \frac{2}{\pi} \arctan \frac{\sqrt{a(b-z)}}{\sqrt{b(z-a)}} - \frac{2}{\pi} \arctan \frac{\sqrt{(c-a)(b-z)}}{\sqrt{(c-b)(z-a)}} + 1 \qquad z \in [a, b]$$

Density: $\rho(z)$ $\qquad \alpha = \frac{1}{10} \qquad C_{crit} = 1.92$



In all we have the following result:

$$\sigma(v,\alpha) = 0 \qquad\qquad v \in [0, v_c(\alpha)]$$

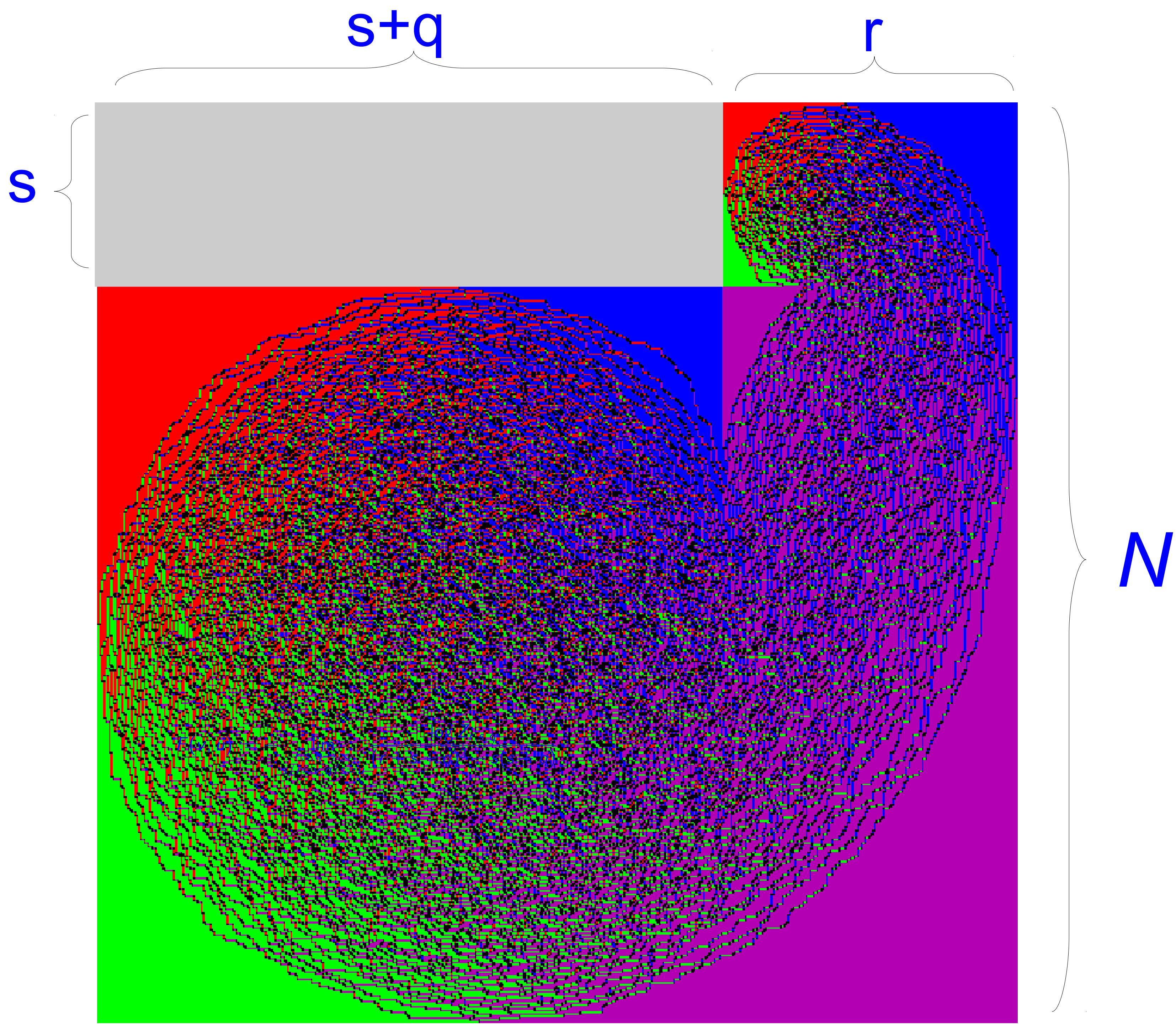
$$\sigma(v;\alpha) = \frac{1}{2}\left[v^2 \log\frac{v^2}{v_c^2} - (1-v)^2 \log\frac{1-v}{1-v_c} - (1+v)^2 \log\frac{1+v}{1+v_c}\right] \qquad v \in [v_c(\alpha), 1]$$

where

$$v_c = v_c(\alpha) = \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}\,.$$

NB1: $v = v_c(\alpha)$ is the value of v corresponding to the Arctic Ellipse

NB2: 3rd order phase transition at $v = v_c$



$N = 500$
$s = 100$

$q = 250$
$r = 150$

$r+s+q=N$

Generic $q \neq 0$ case:

Potential $V(\mu)$ is not linear any more:

$$V(\mu) = -\mu \log \alpha + \mu \log \mu - (\mu + u) \log(\mu + u) + u \log u, \qquad u := \frac{q}{s}$$

bulky calculations, but everything remains qualitatively the same:

Two scenarios; in scenario i) we get the value

$$b = \frac{(1 + \sqrt{\alpha(1+u)})^2}{1 - \alpha}$$

for the right end-point.
Equating $b = c$ and expressing the result in terms of cartesian coordinates:

$$c := \frac{r}{s} = \frac{1-x}{y}, \qquad u := \frac{N-r-s}{s} = \frac{x-y}{y}$$

we readily recover the Arctic Ellipse.

Moreover, when the tip of the rectangle reaches the position of the Arctic Ellipse in the original, plain Aztec Diamond, a 3rd order phase transition occurs.

- When the tip of the rectangle reaches the position of the Arctic Ellipse in the original, plain Aztec Diamond, a 3rd order phase transition occurs.

- As we vary r, s, q , the tip of the rectangle, never touches the Arctic Circle, but repels it away.

- Due to the correspondence between tilings of Aztec diamond and non-intersecting lattice paths , similar phenomena should be observed when you constrain the lattice paths under a `bridge',

- and also when you constrain a set of non-intersecting brownian motion or vicious walkers (watermelon) through a slit.

A new universality class?