The nineteen-vertex model: from supersymmetry to combinatorics

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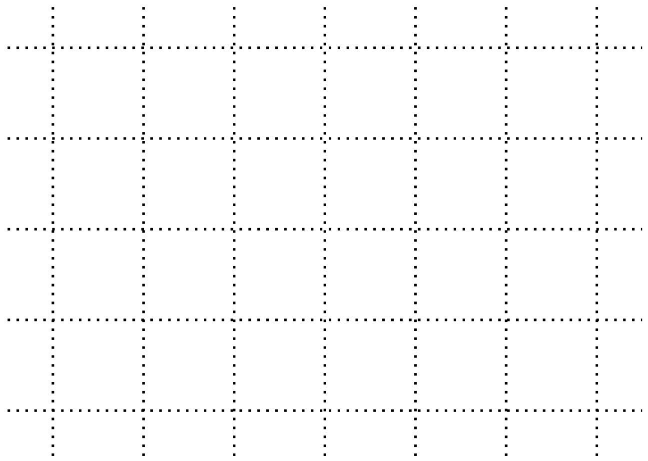

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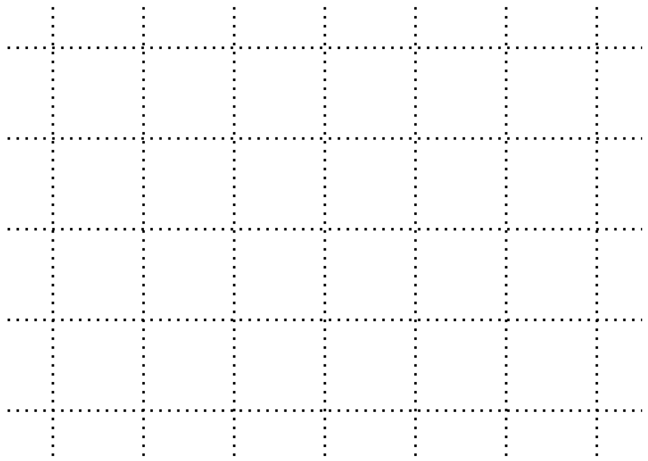

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- At every vertex the number of outgoing arrows equals the number of ingoing arrows.

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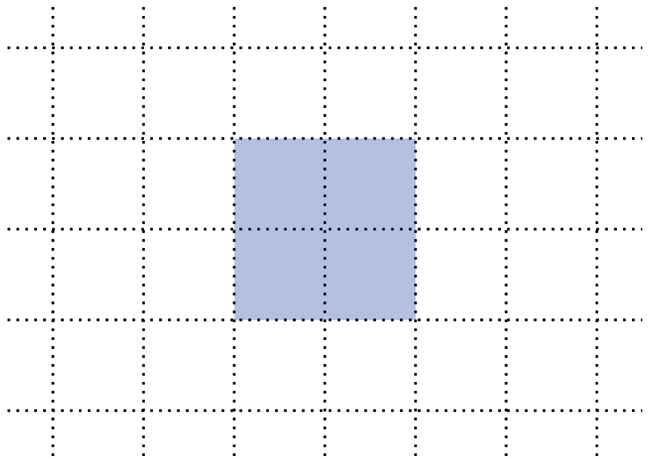

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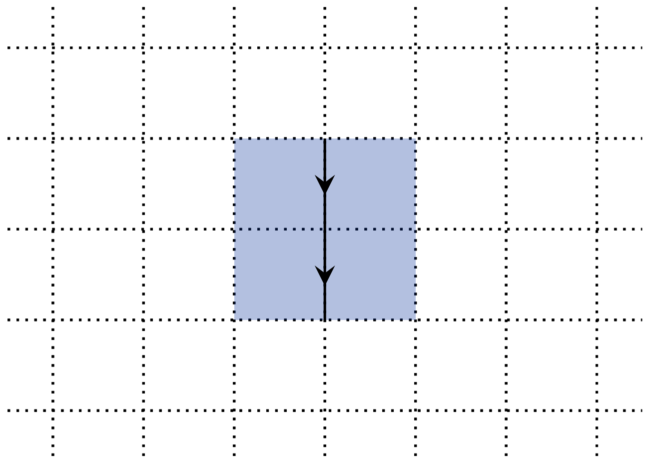

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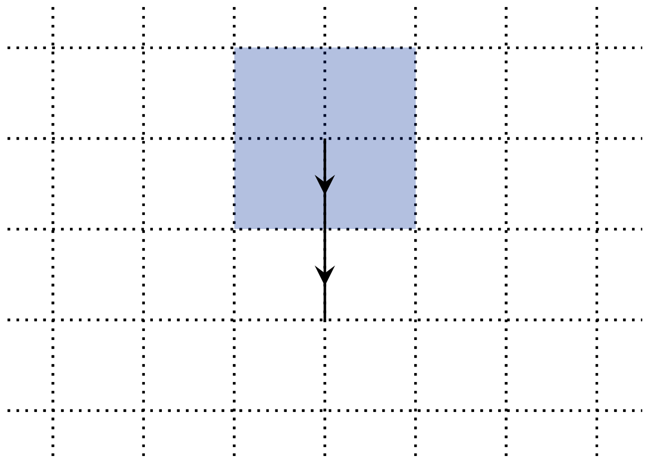

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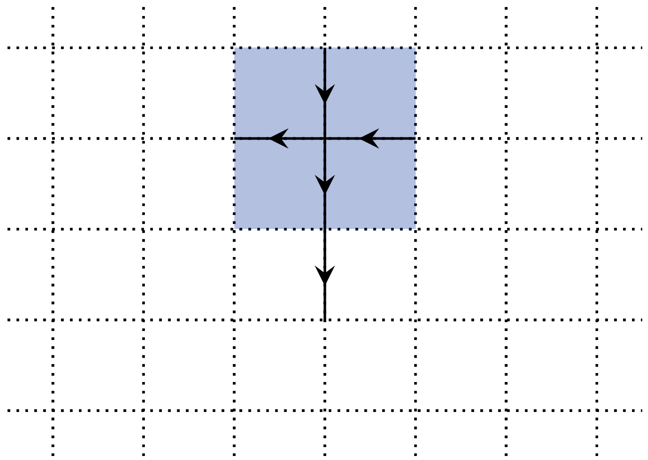

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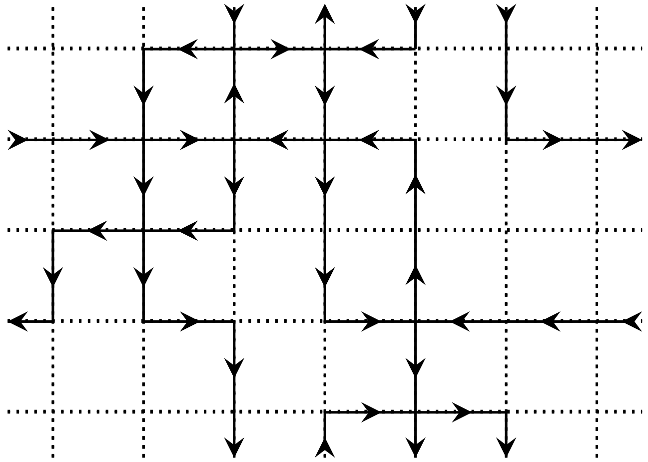

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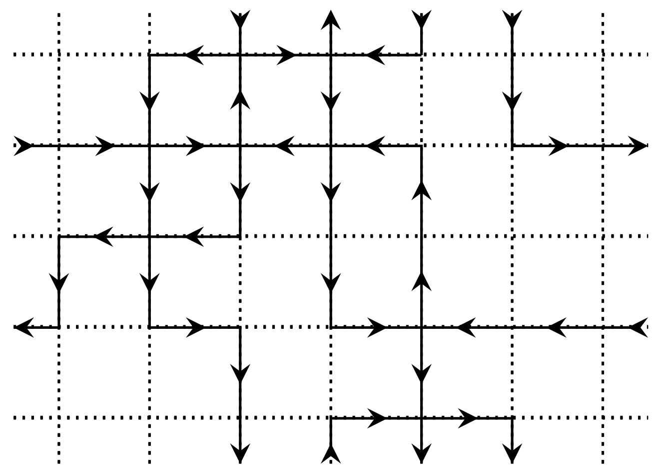

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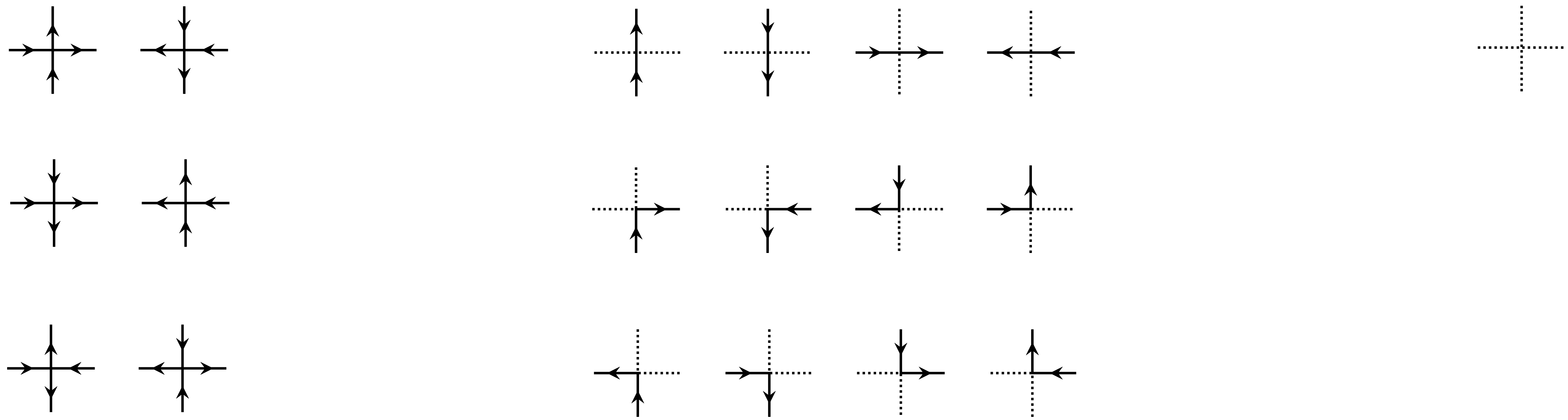
19 admissible configurations around a vertex



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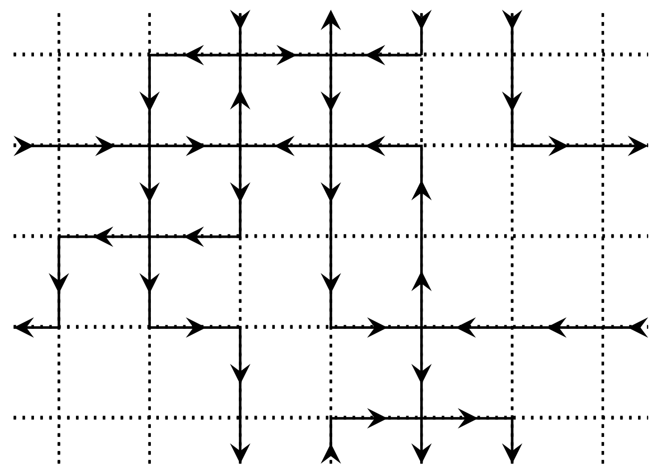

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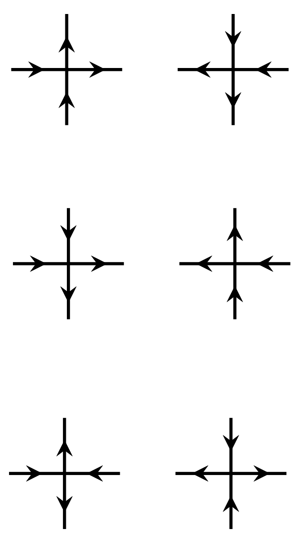
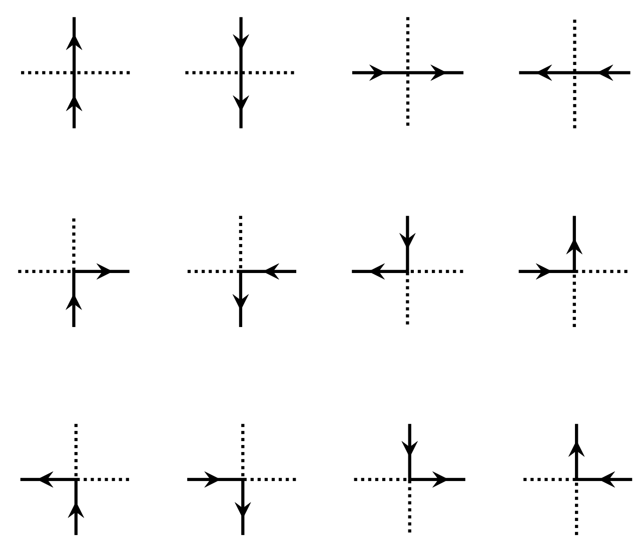
19 admissible configurations around a vertex with statistical weights



$a = [qw][q^2w]$

$b = [w][qw]$

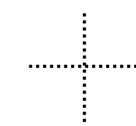
$c = [q^{-1}w][w]$



$x = [q^2][qw]$

$y = [q^2][w]$

$z = [q^2][q]$



$d = b + z$

$[w] = w - w^{-1}$

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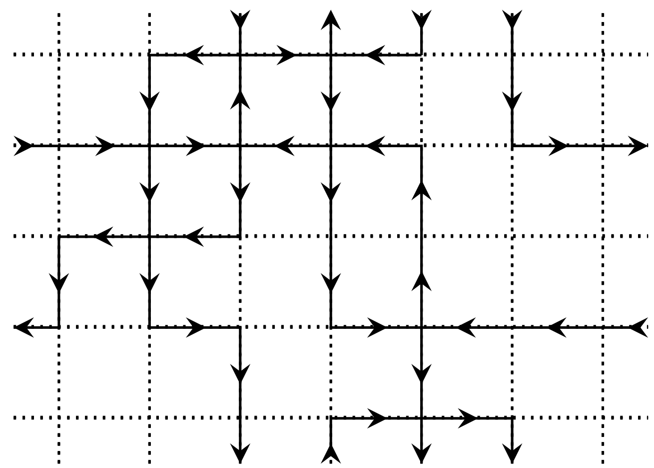

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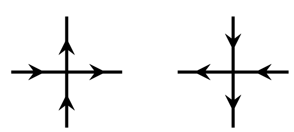
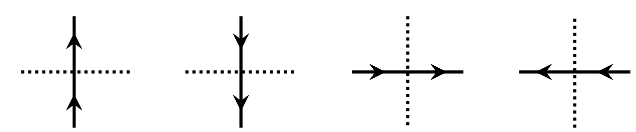
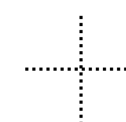
 $a = [qw][q^2 w]$  $x = [q^2][qw]$ 

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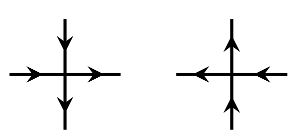
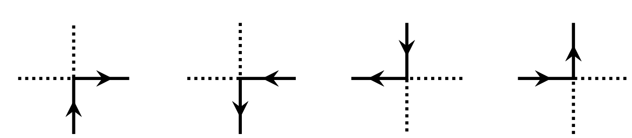
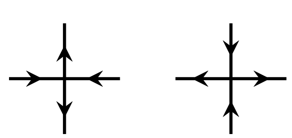
 $b = [w][qw]$  $y = [q^2][w]$

 $c = [q^{-1}w][w]$  $z = [q^2][q]$

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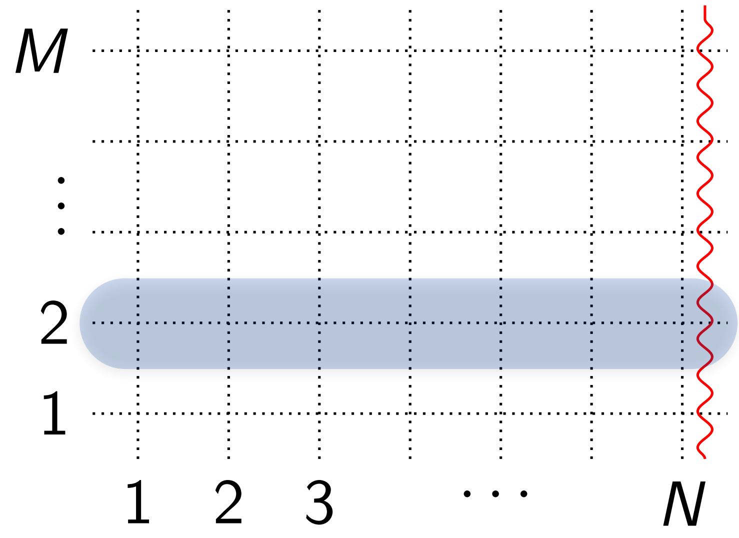
Weights solve the Yang-Baxter equation (Zamolodchikov & Fateev `81).

The transfer matrix



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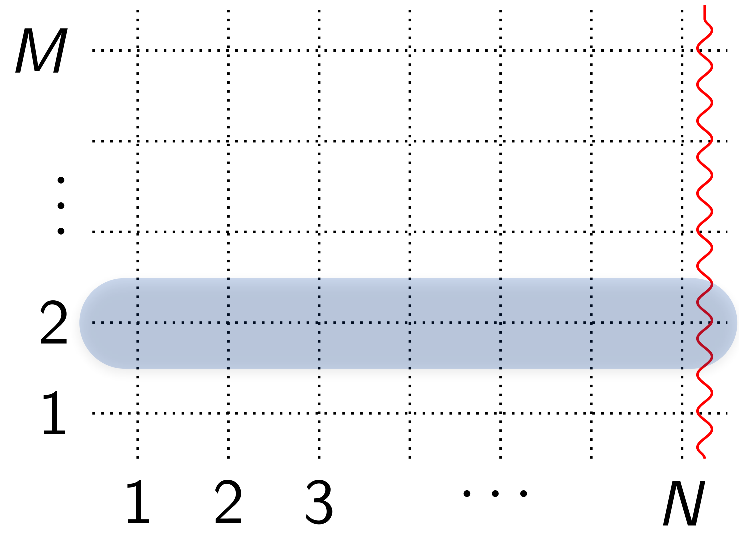

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- Allow for a *defect/twist* measuring state of last horizontal edge.

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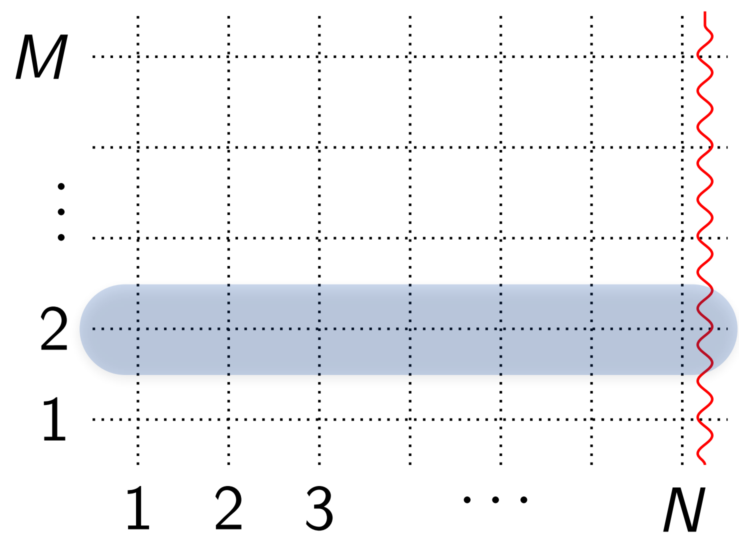

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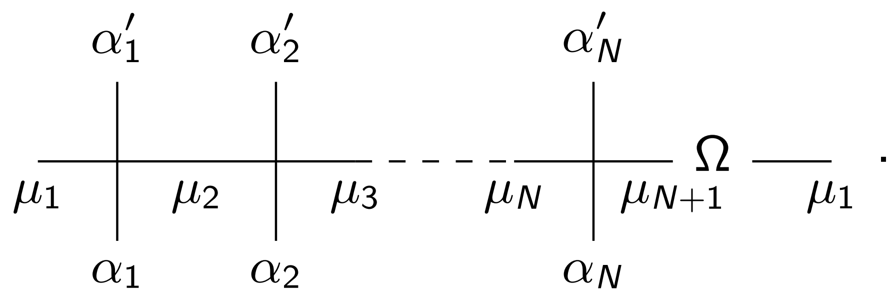
$$T_{\{\alpha'\},\{\alpha\}}(w) = \sum_{\substack{\mu_1,\dots,\mu_{N+1}\\ =\uparrow,0,\downarrow}}$$



Matrix elements of an operator $T(w)$ on the space $V^{\otimes N}$, $V = \mathrm{span}\{e_\uparrow, e_0, e_\downarrow\}$

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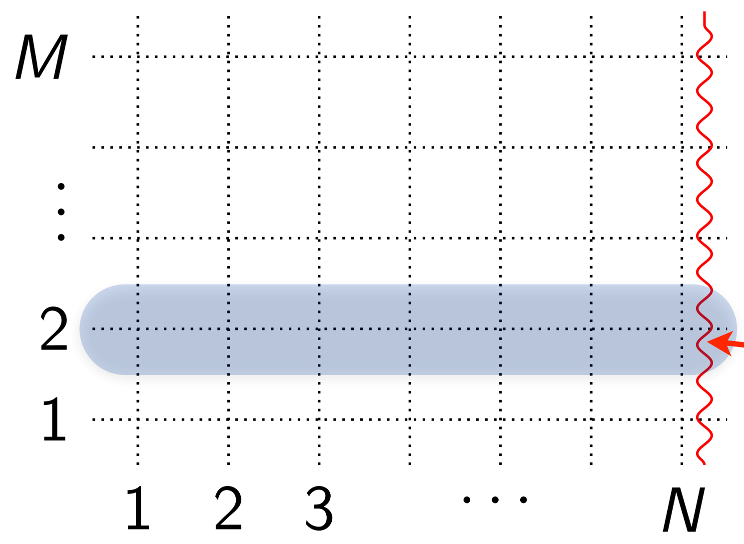

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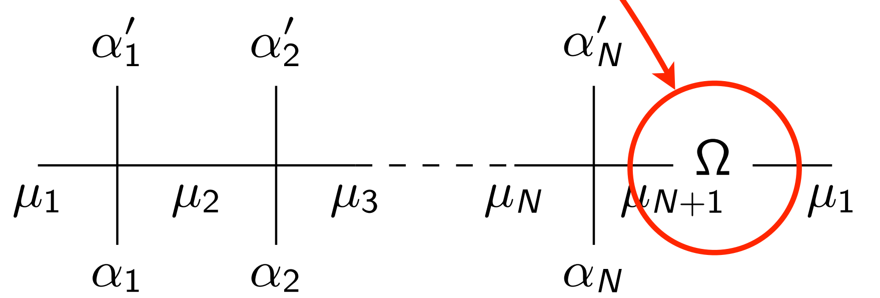

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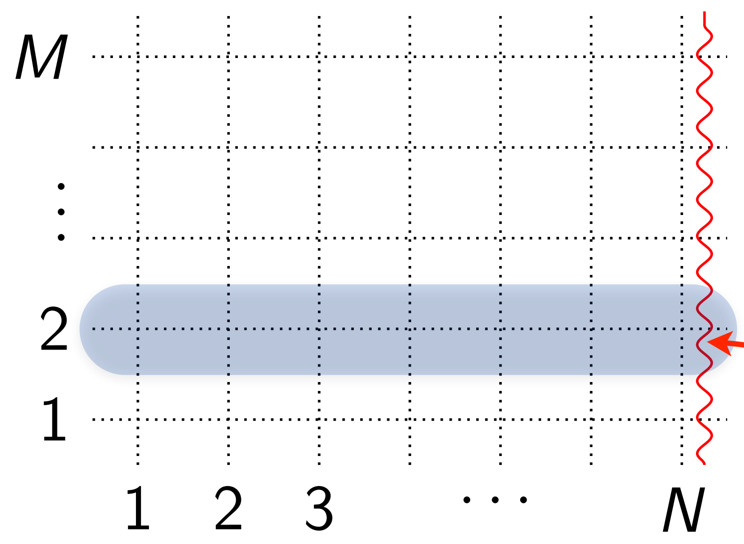

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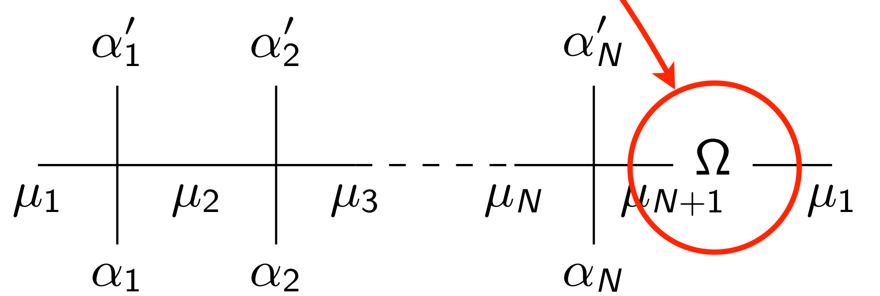

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Commuting transfer matrices $[T(w), T(w')] = 0$

because of *(1)* YBE and *(2)* choice of boundary conditions

Diagonal twist $\Omega_+ = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$

Non-diagonal twist $\Omega_- = -\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$

Spin-1 XXZ chain

$$T(w) = \text{const.} \times S \left(1 + (w-1) \left(\frac{2}{[q^2]} H - N \right) + \ldots \right)$$

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$$H = \sum_{j=1}^{N}\left(\sum_{a=1}^{3} J_a (s_j^a s_{j+1}^a + 2(s_j^a)^2) - \sum_{a,b=1}^{3} A_{ab} s_j^a s_j^b s_{j+1}^a s_{j+1}^b\right)$$

$$J_1 = J_2 = 1,\ J_3 = \frac{1}{2}(q^2 + q^{-2}), \quad A_{12} = 1,\ A_{13} = A_{23} = q + q^{-1} - 1,\ A_{aa} = J_a.$$

- Spin-1 representation of $\mathfrak{su}(2)$: $s^a,\ a = 1, 2, 3$
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Dynamical supersymmetry

$$H = \{\mathfrak{Q}, \mathfrak{Q}^\dagger\} = \mathfrak{Q}\mathfrak{Q}^\dagger + \mathfrak{Q}^\dagger\mathfrak{Q}, \quad \mathfrak{Q}^2 = (\mathfrak{Q}^\dagger)^2 = 0$$

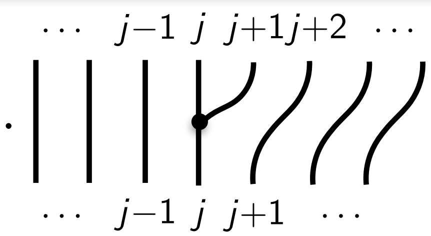
$$[N, \mathfrak{Q}] = \mathfrak{Q}$$

$$[H, \mathfrak{Q}] = [H, \mathfrak{Q}^\dagger] = 0$$

Supercharge

$$\mathfrak{Q} = \sqrt{\frac{N}{N+1}} \sum_{j=0}^{N} (-1)^j \mathfrak{q}_j \qquad \begin{cases} S\mathfrak{q}_j S^{-1} = \mathfrak{q}_{j+1}, & j = 0, \ldots, N-1 \\ S\mathfrak{q}_N = \mathfrak{q}_0 \end{cases}$$

Local splitting operation $\mathfrak{q} : V \to V \otimes V$

$\mathfrak{q}_j$ $\cdots$  $\cdots$

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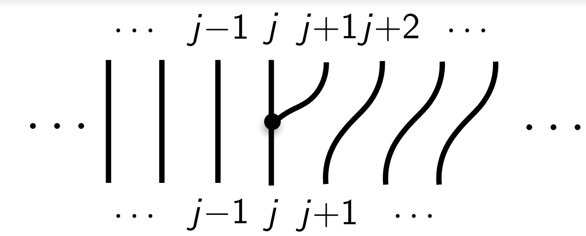

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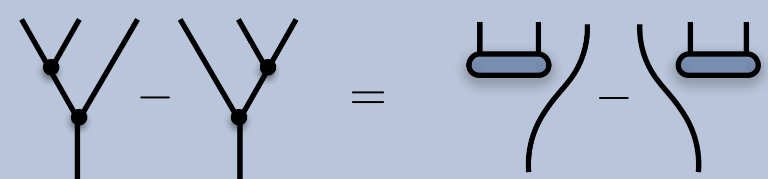

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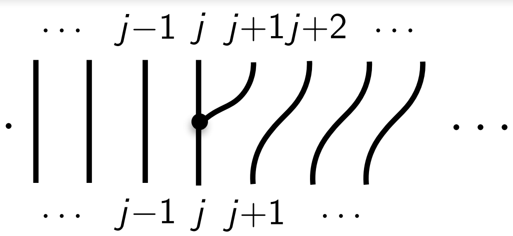

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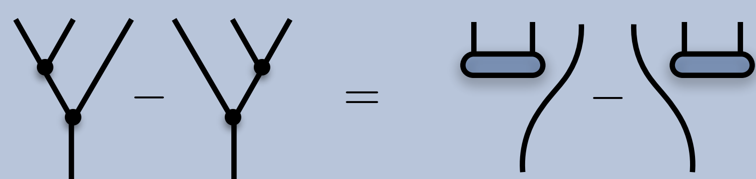

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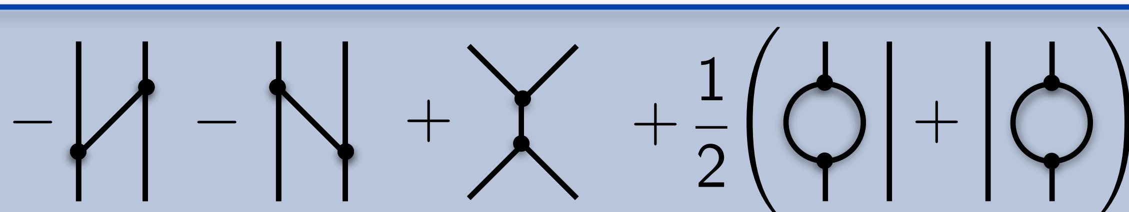
Hamiltonian $H = \{\mathfrak{Q}, \mathfrak{Q}^\dagger\} = \sum_{j=1}^{N-1} \mathfrak{h}_{j,j+1} + \tilde{\mathfrak{h}}_{N,1}$

$\mathfrak{h} =$



Ground state

In the present case $$\begin{cases} \mathfrak{q} e_{\downarrow} &= \frac{1}{2}(q+q^{-1})(e_0 \otimes e_{\downarrow} - e_{\downarrow} \otimes e_0) \\ \mathfrak{q} e_0 &= e_{\uparrow} \otimes e_{\downarrow} - e_{\downarrow} \otimes e_{\uparrow} \\ \mathfrak{q} e_{\uparrow} &= \frac{1}{2}(q+q^{-1})(e_{\uparrow} \otimes e_0 - e_0 \otimes e_{\uparrow}) \end{cases}$$

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In this talk: square norm $||\Phi||^2 = \sum_\alpha \Phi_\alpha^2$

Ground state

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For the twists $\Omega_\pm$ the FZ-Hamiltonian has an ***exact zero-energy ground state*** $H\Phi = 0$ for every N in the subspaces where $S \equiv (-1)^{N+1}$ at ***any value of the anisotropy parameter q***.

Supersymmetry singlet: $\mathfrak{Q}\Phi = 0, \quad \mathfrak{Q}^\dagger\Phi = 0, \quad \Phi \in V_N.$

Components: $\Phi = \sum_\alpha \Phi_\alpha e_{\alpha_1} \otimes \cdots e_{\alpha_N}, \quad \Phi_\alpha = \Phi_\alpha(x), \quad x = \frac{1}{2}(q+q^{-1})$

- Polynomials in x with integer coefficients ***in a suitable normalisation***.
- Some coincide with generating functions for ***weighted enumeration of alternating sign matrices***.

In this talk: square norm $||\Phi||^2 = \sum_\alpha \Phi_\alpha^2$

Diagonal twist and weighted enumeration of ASMs

Exact determination of eigenvector for small N:

$$||\Phi||^2 = A_N\left(t = (q+q^{-1})^2\right)$$

with the generating function

$$A_N(t) = \sum_{N\times N\text{ ASMs}} t^k$$

k : number of minus signs in given ASM

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Case of 3 x 3 alternating sign matrices

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$
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$$A_3(t) = 6 + t$$

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Exact formula for $A_N(t)$ at any N known.

Non-diagonal twist: another symmetry class

Exact determination of eigenvector for small N:

$$||\Phi||^2 = \frac{A_{2N}^{\mathrm{HT}}(t)}{A_N(t)}, \quad t = (q+q^{-1})^2$$

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HTASM = **H**alf-**T**urn symmetric **A**lternating **S**ign **M**atrix

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For $N = 3$: $A_{2N}^{HT}(t) = (6+t)(8+10t+2t^2)$

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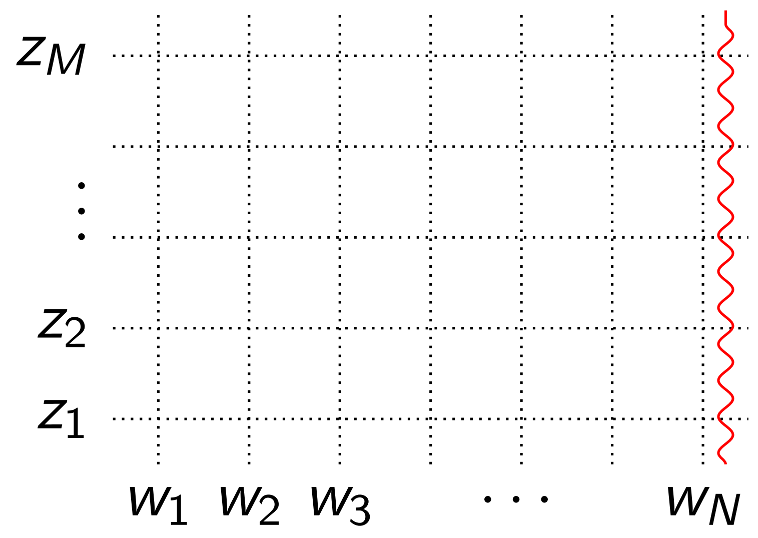
Introducing inhomogeneities



- Weight of a vertex at a site depends on the ***ratio***
  $$z_i / w_j$$
- ***Inhomogeneous*** transfer matrix
  $$T(w) = T(w | w_1, \dots, w_N)$$

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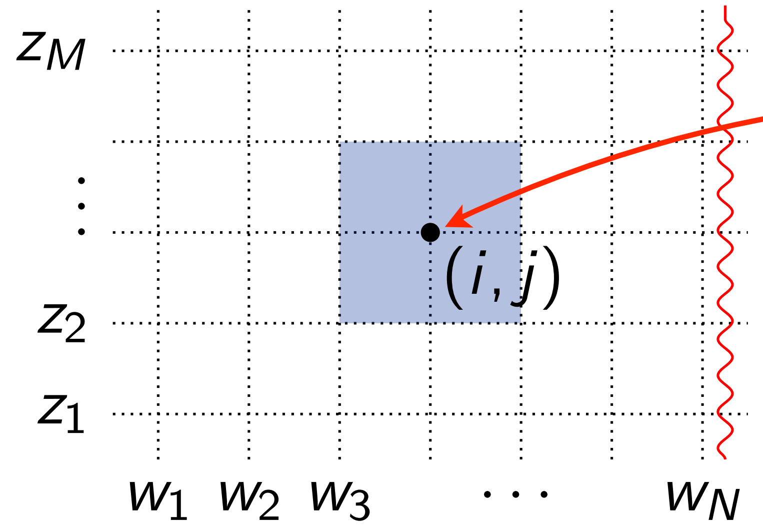

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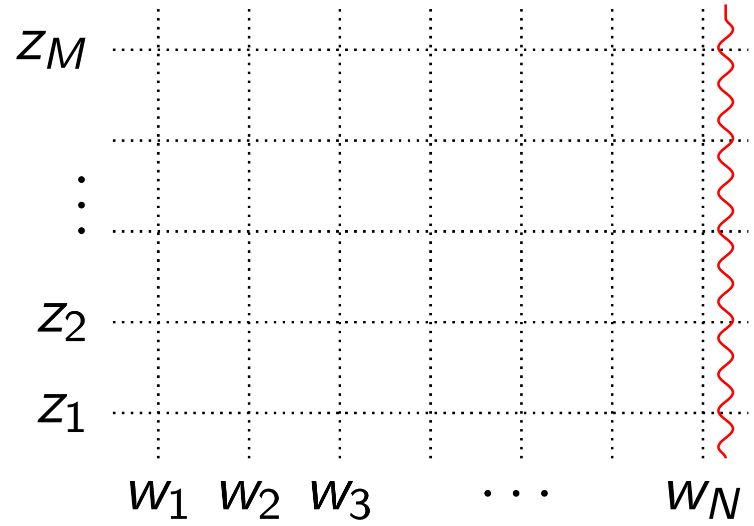

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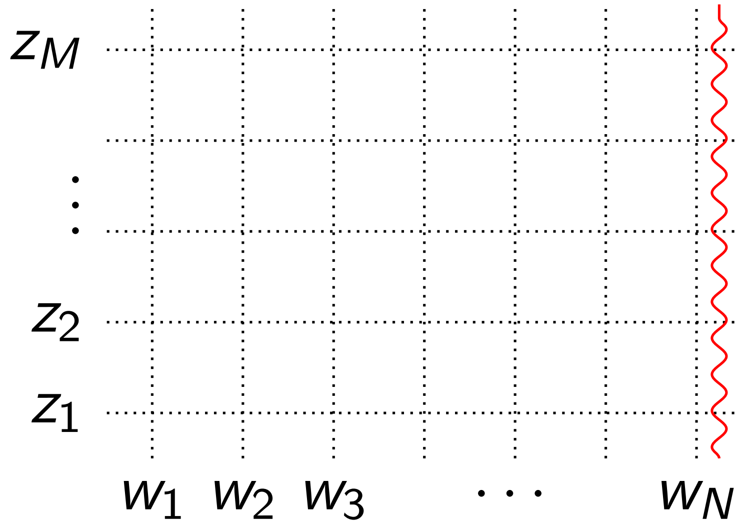

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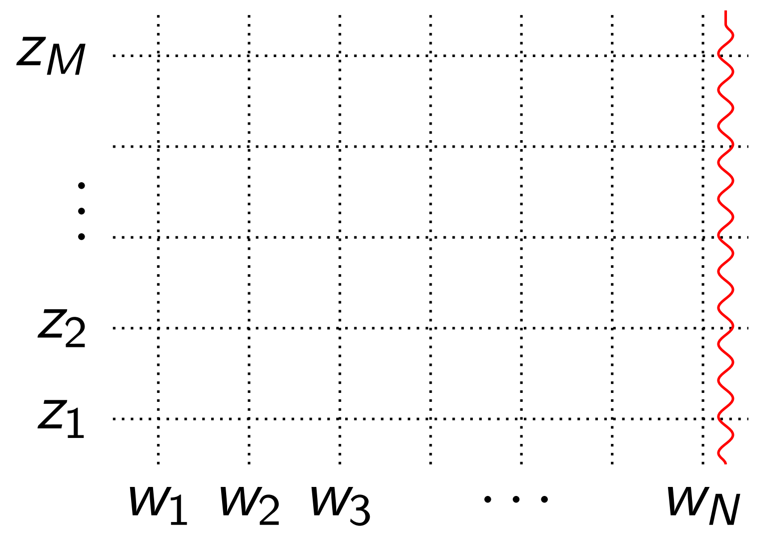

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$$\Lambda = \pm\prod_{j=1}^{N}\left[\frac{q^{-1}w_j}{w}\right]\left[\frac{q^{2}w_j}{w}\right]$$

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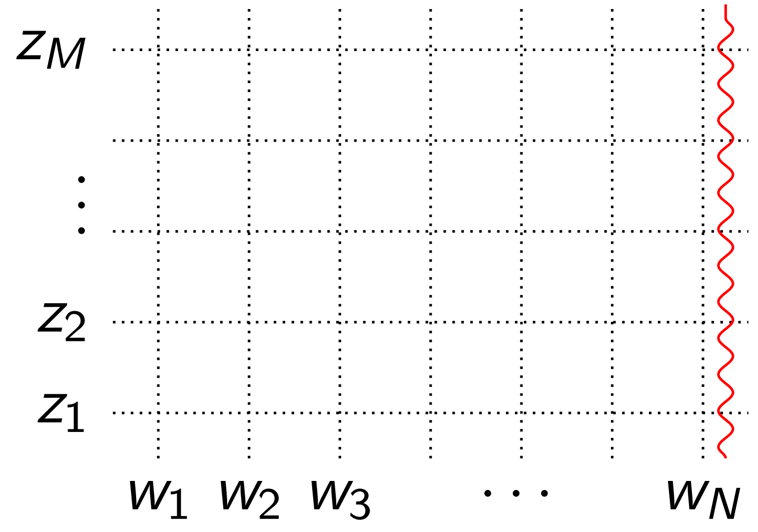

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Conjecture 2: In ***minimal polynomial normalisation*** the corresponding eigenvector is a homogeneous polynomial of degree $d_N = N(N-1)$

$$\Phi(\lambda w_1, \dots, \lambda w_N) = \lambda^{d_N}\Phi(w_1, \dots, w_N)$$

Quadratic sum rule for the diagonal twist

For the diagonal twist we have the quadratic sum rule

$$\sum_{\alpha} \Phi_\alpha(w_1^{-1}, \dots, w_N^{-1})\Phi_\alpha(w_1, \dots, w_N) = Z_N^{\mathrm{IK}}(w_1, \dots, w_N; w_1, \dots w_N)$$

With the partition function of the ***6-vertex model*** on an *N x N* square with ***domain-wall boundary conditions***

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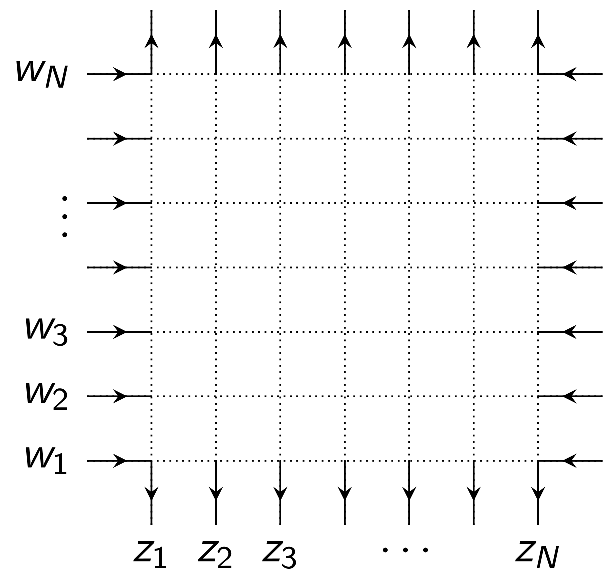

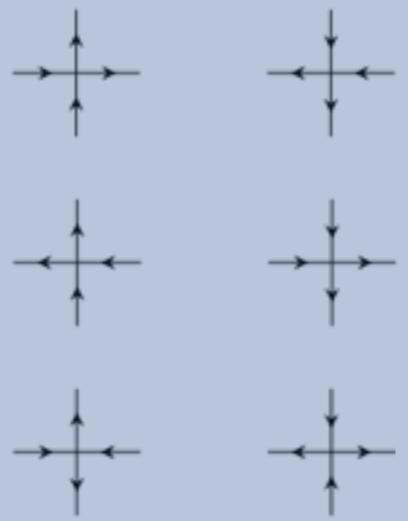

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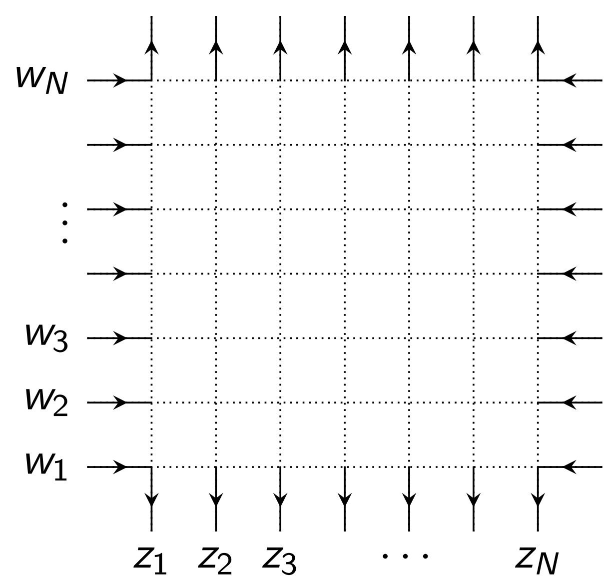

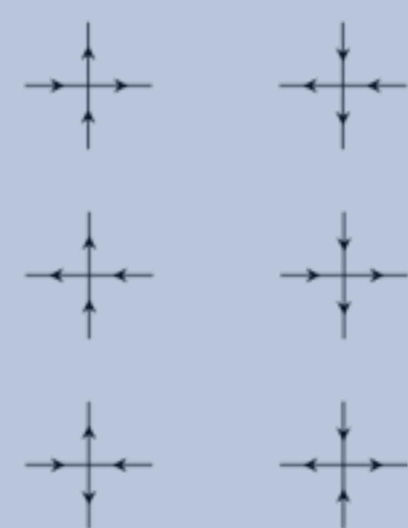

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$$Z_N^{\mathrm{IK}}(\{z\},\{w\}) = \frac{\prod_{i,j=1}^{N} \mathfrak{a}(z_i/w_j)\mathfrak{b}(z_i/w_j)}{\prod_{1\le i<j\le N}[z_i/z_j][w_i/w_j]} \times \det_{i,j=1,\dots,N}\left(\frac{\mathfrak{c}(z_i/w_j)}{\mathfrak{a}(z_i/w_j)\mathfrak{b}(z_i/w_j)}\right)$$

Conjecture for the spin-reversal twist

For the spin-reversal twist the we have the quadratic sum rule

$$\sum_{\alpha} \Phi_\alpha(w_1^{-1}, \dots, w_N^{-1}) \Phi_\alpha(w_1, \dots, w_N) = \frac{Z_N^{\mathrm{HT}}(w_1, \dots, w_N; w_1, \dots w_N)}{Z_N^{\mathrm{IK}}(w_1, \dots, w_N; w_1, \dots w_N)}$$

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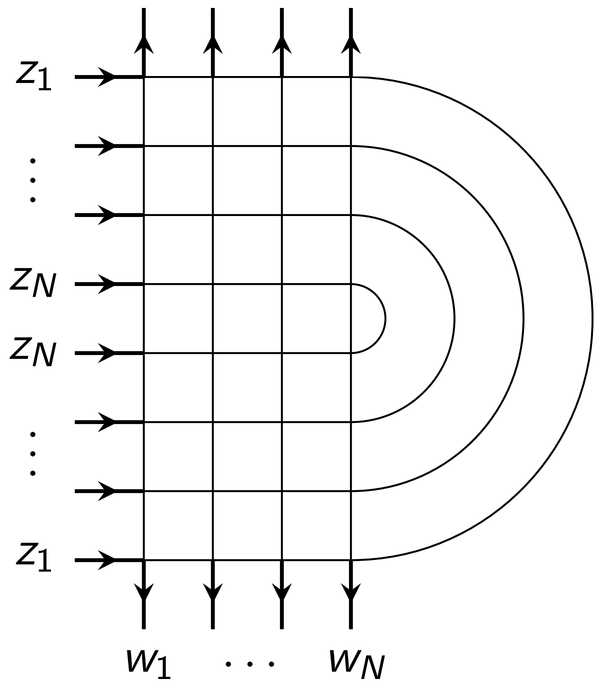
Partition function of the ***six-vertex model*** with ***half-turn boundary conditions***



Partition function ratio **(Kuperberg)**

$$\frac{Z_N^{\mathrm{HT}}(\{z\},\{w\})}{Z_N^{\mathrm{IK}}(\{z\},\{w\})} = \frac{\prod_{i,j=1}^N \mathfrak{a}(z_i/w_j)\mathfrak{b}(z_i/w_j)}{\prod_{i,j=1}^N [z_i/z_j][w_i/w_j]} \det_{i,j=1,\dots,N} M_{ij}^{\mathrm{HT}}$$

$$M_{ij}^{\mathrm{HT}} = \frac{1}{\mathfrak{a}(z_i/w_j)} + \frac{1}{\mathfrak{b}(z_i/w_j)}$$

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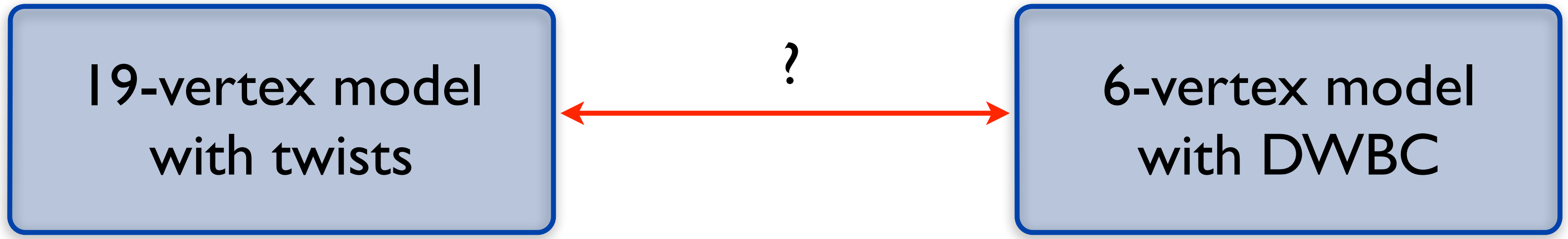

Conclusion

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- Ground states and weighted enumeration of alternating sign matrices.
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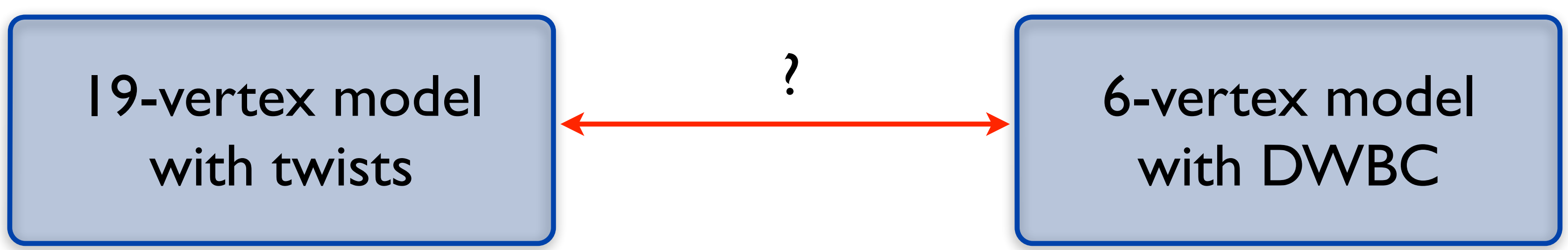

- *Elliptic extension: 41-vertex model*