Combinatorial points of the dilute loop model?

Combinatorial Physics, Cardiff

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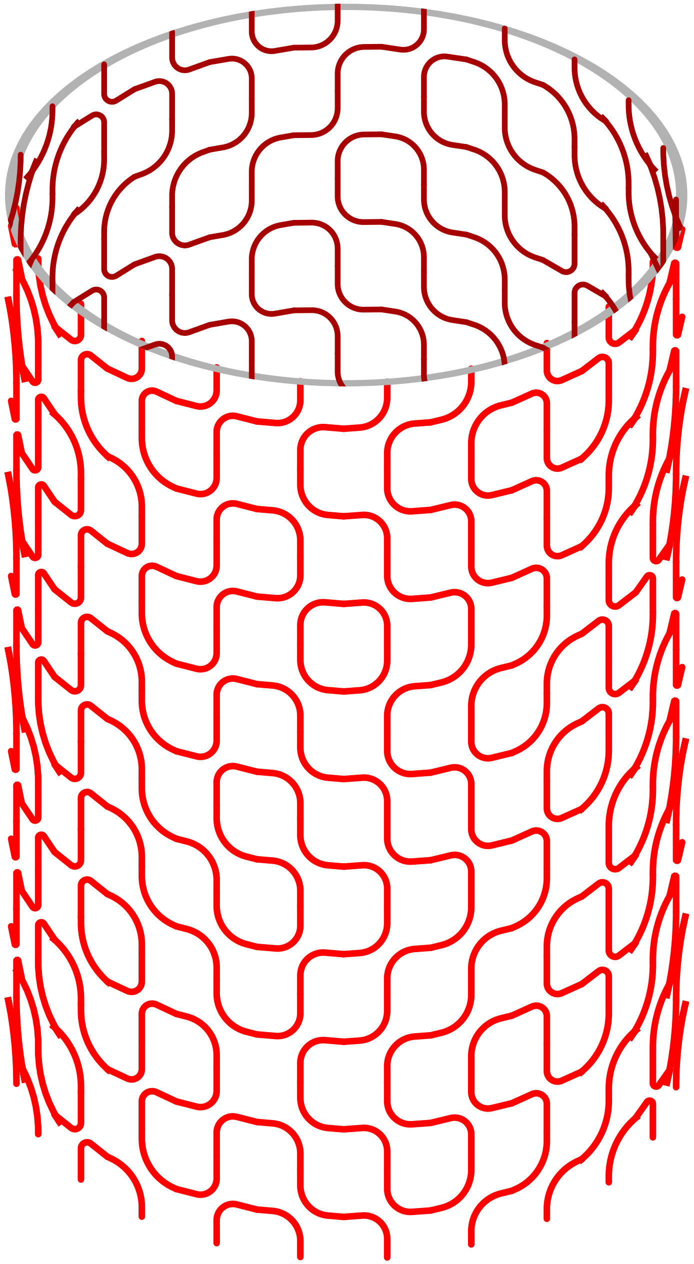
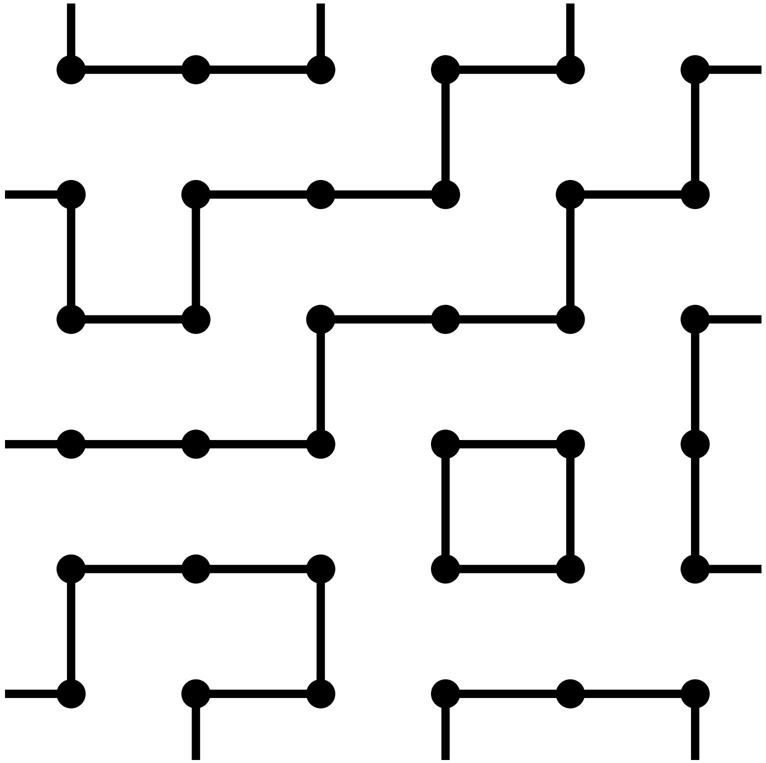
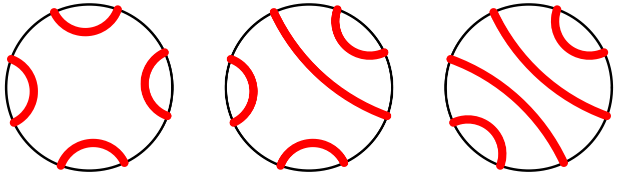




Attempt to find a generalization of the Cantini-Sportiello Theorem alias Razumov-Stroganov conjecture and similar counts.

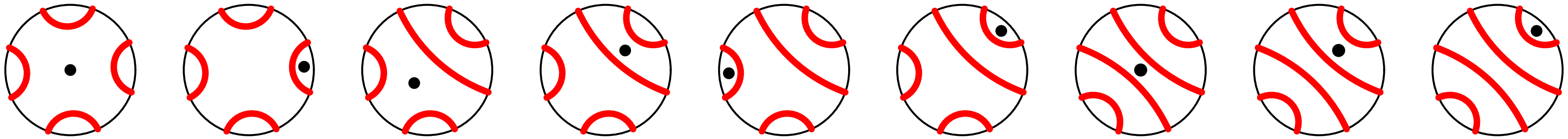


multiplicity	2	8	4
weight.	7	3	1

total: 42

The weights for size $L = 2n$ correspond to the number of FLP patterns on a $n \times n$ grid with the same matching.

The well-known sequence 1, 2, 7, 42, 429, 7436, . . .

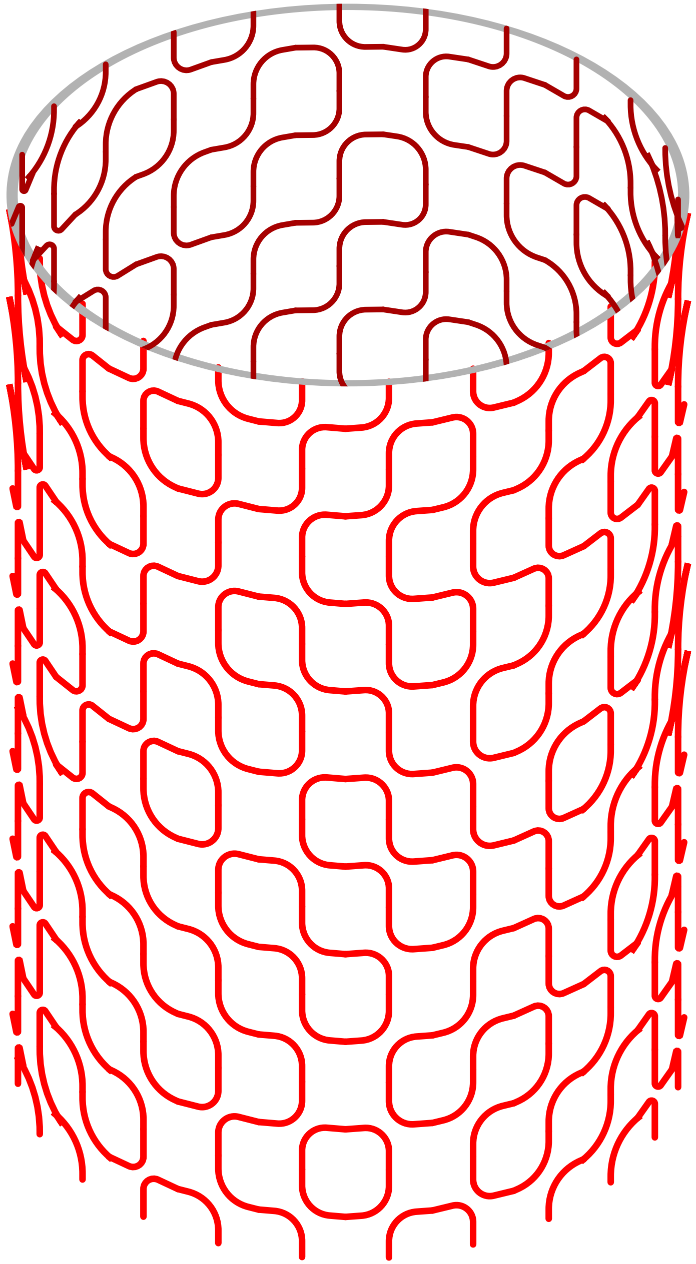


mult.	2	8	8	8	16	8	4	8	8
wgt.	588	84	216	29	20	7	72	29	1

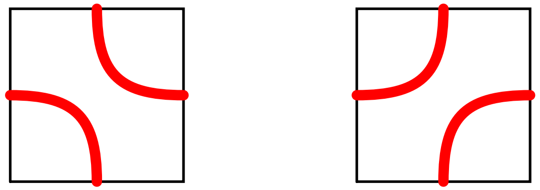
total: 5544

In this case the weights correspond to the number of half-turn symmetric FPL patterns on a $L \times L$ grid,

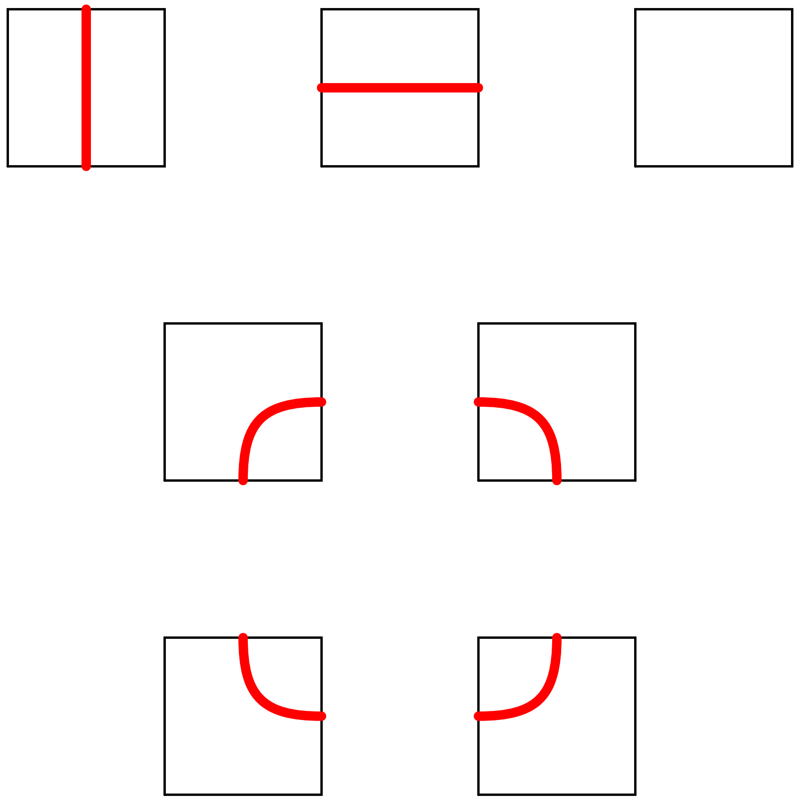
i.e. the sequence 1, 2, 3, 10, 25, 140, 588, 5544, 39204, . . .



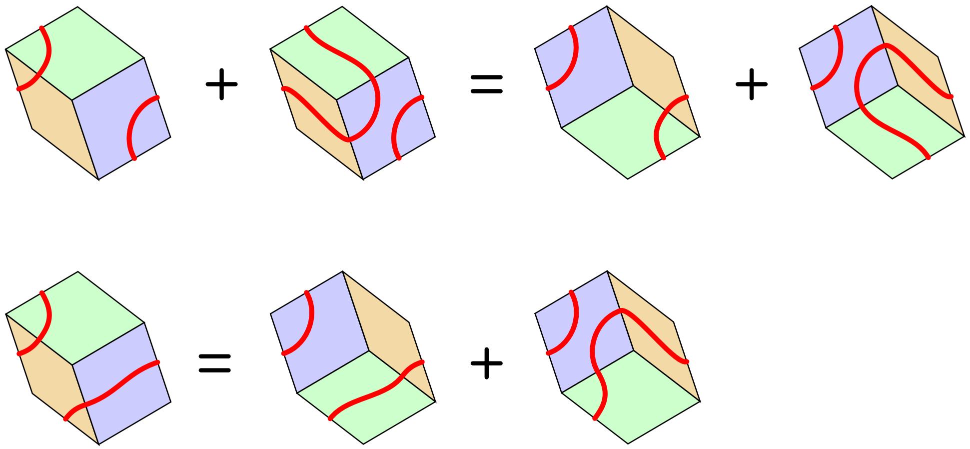
On the cylinder we allow not only for the configurations:

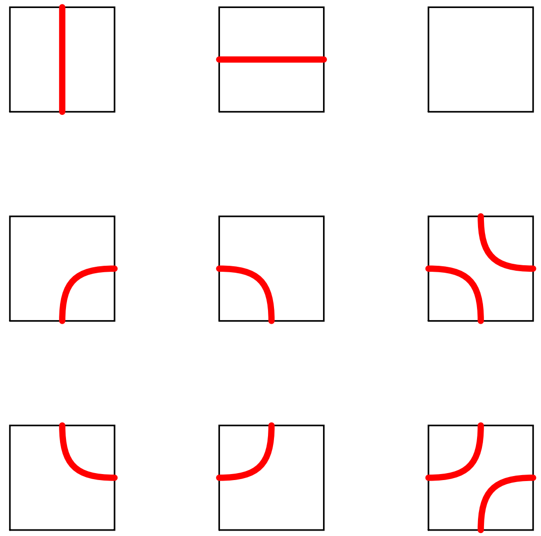


but also for the following



Assume the weights solve the YB-eq.



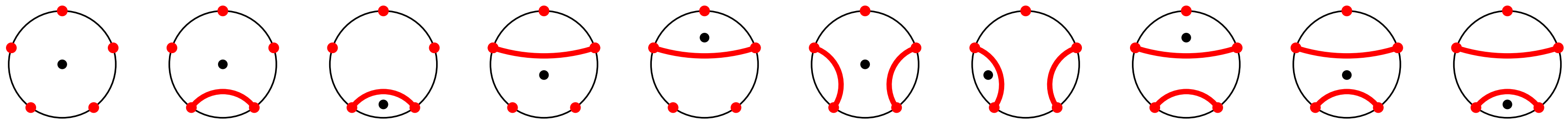


And the eigenvalue of the transfer matrix should be purely exponential in the system size.

Four cases known:

loop weight	universality class
1	percolation
0	Self-avoiding-walk
0	Interacting-SAW
0	θ-SAW

We will focus on the percolation case first.



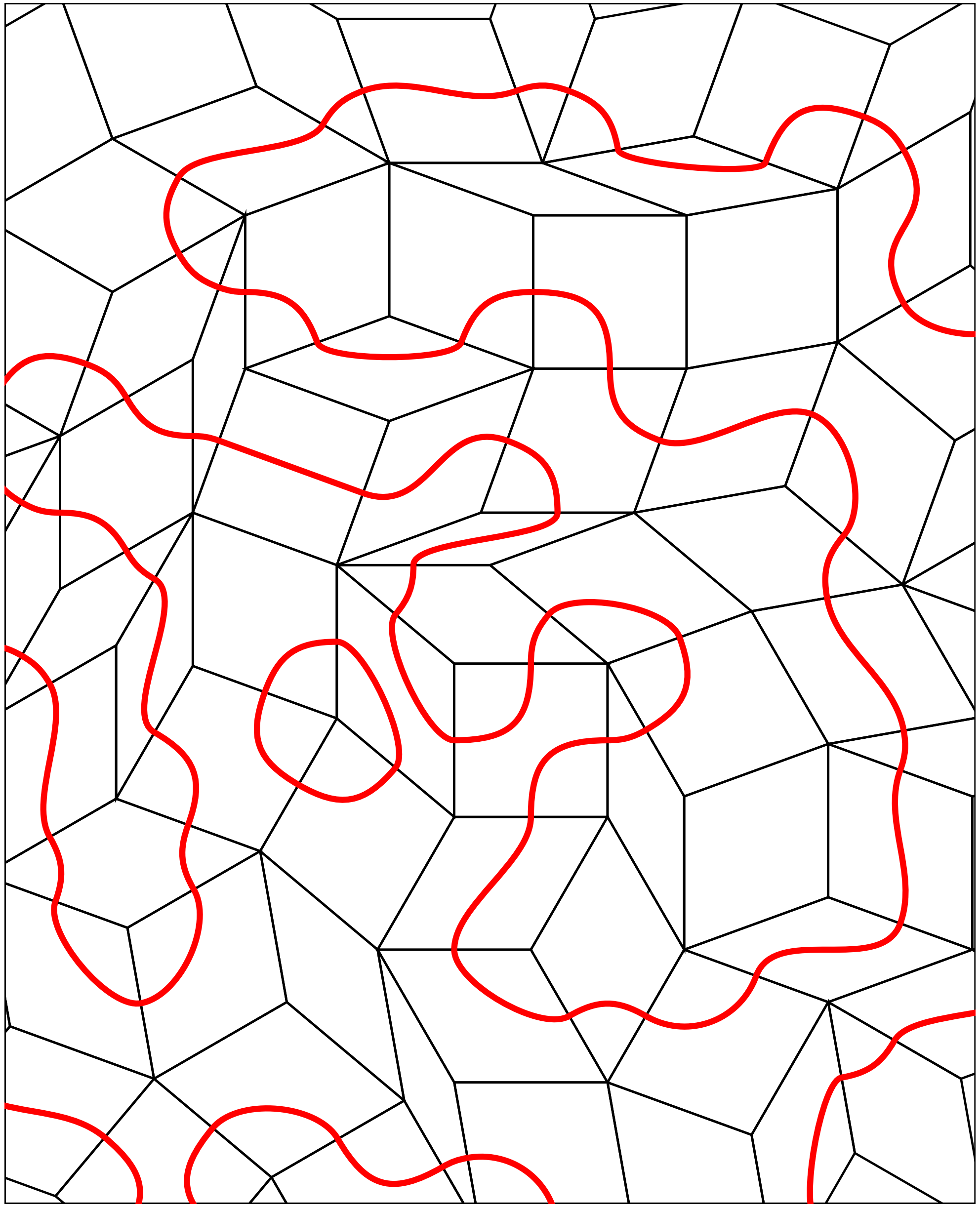
wgt.	275	171	104	148	127	94	28	33	76	16

The sum of all weights is $2^{L-1}\times$ the weight of the empty element.

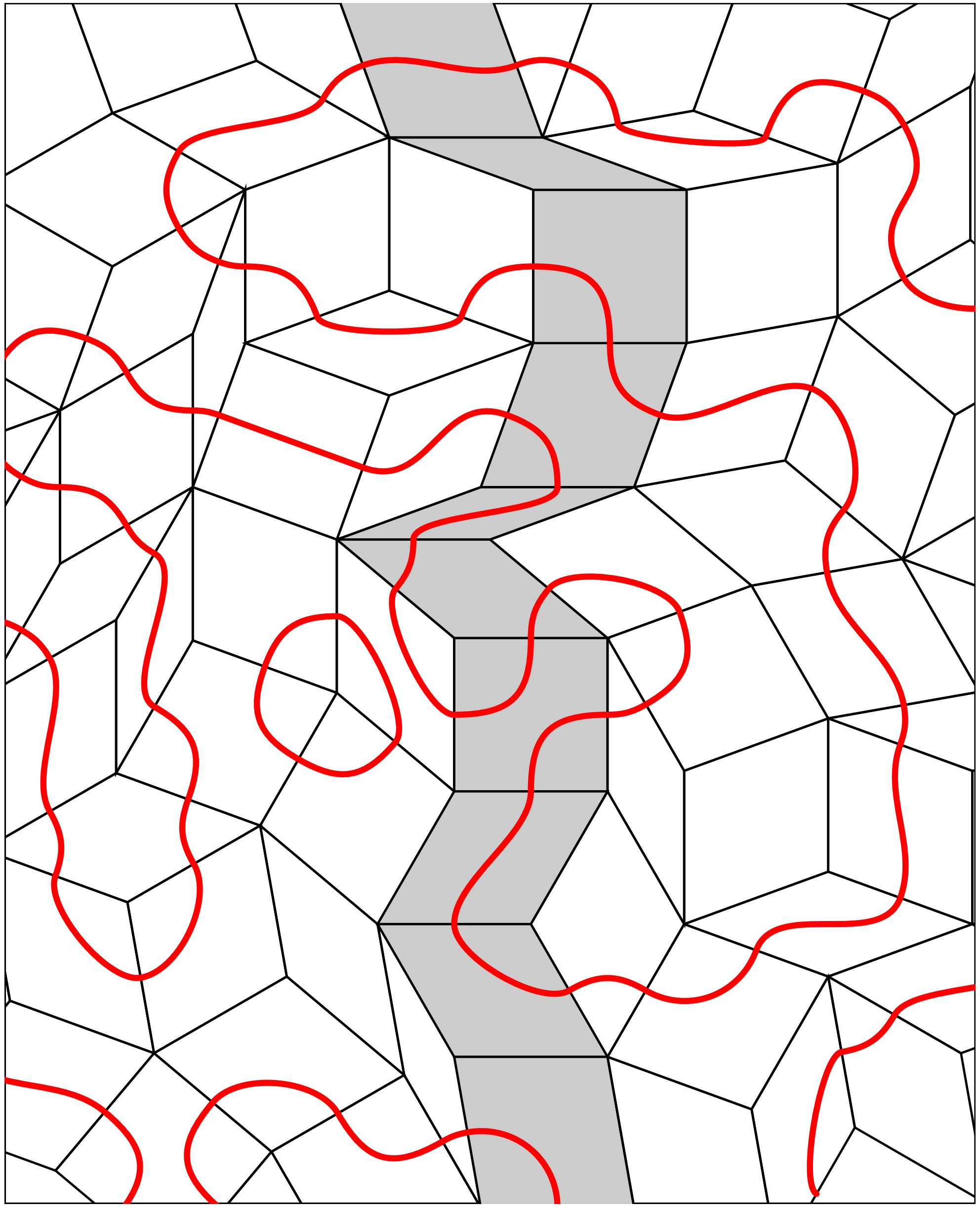
L	weight of empty element
1	1
2	2
3	9
4	32
5	275
6	5760
7	98441
8	4128768
9	425662371
10	35997491200

Do these numbers also have a combinatorial meaning?

Can we find an expression for them?

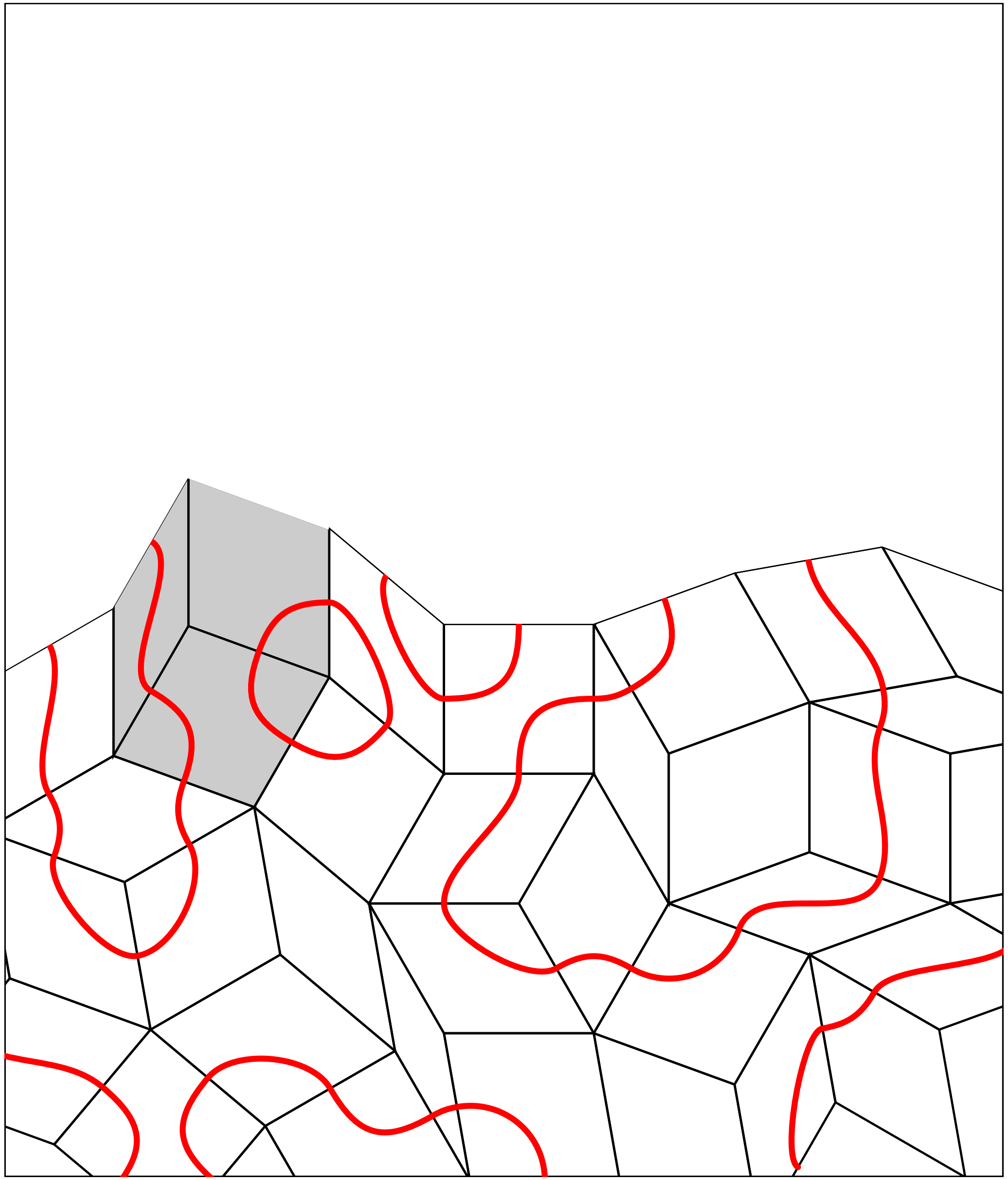


When we make the model inhomogeneous we can make explicit use of the YBeq.



To make connection with yesterdays lectures, the parameter lines are represented by paths of rhombi with parallel sides.

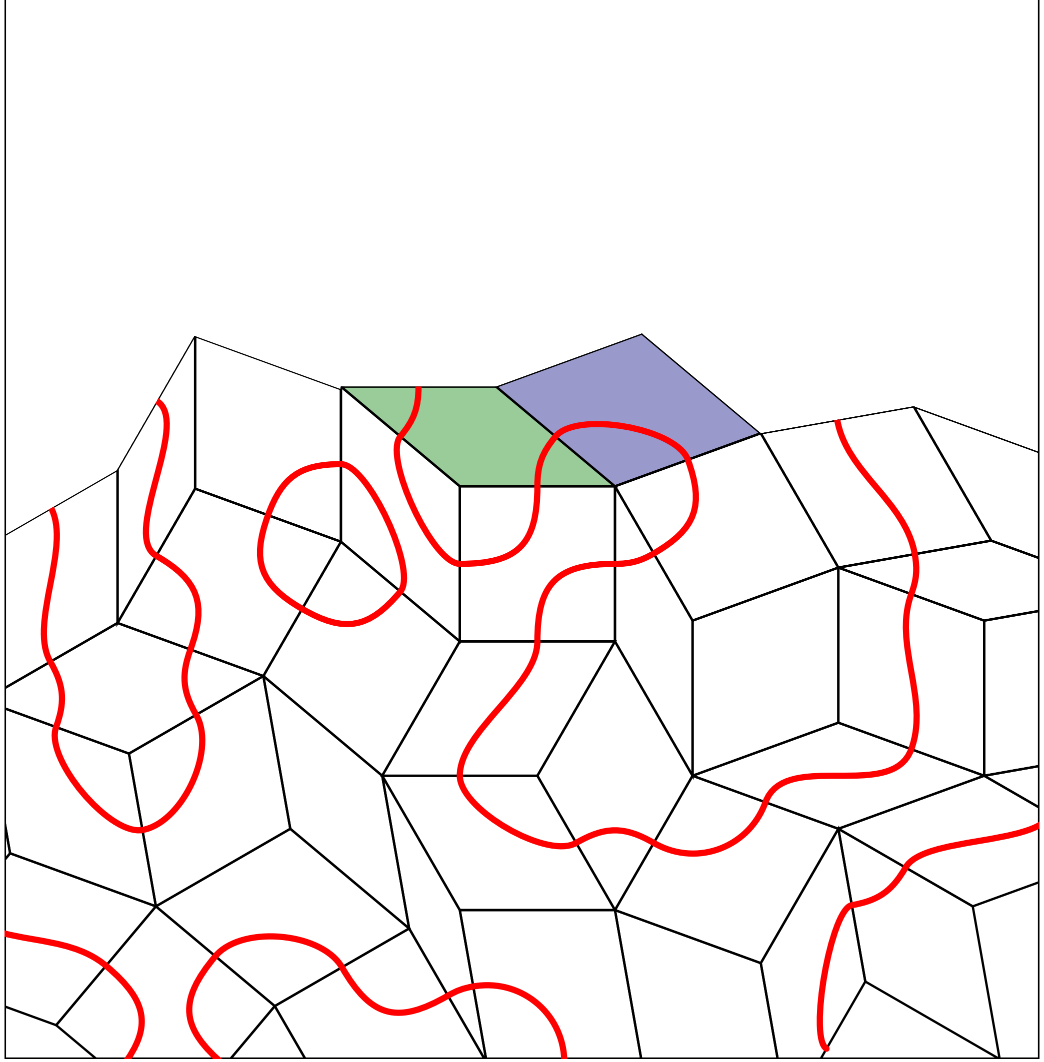
In fact the orientation of the parallel sides represents the value of the parameter.



Suppose a cylinder is tiled up to some border.

Then the weights of the loop configurations induce a probability distribution Ψ for the (partial) matching of the border edges.

From the YBeq it follows that Ψ does not depend on the specific tiling, only on the geometric shape of the border, i.e. the sequence of oriented edges.

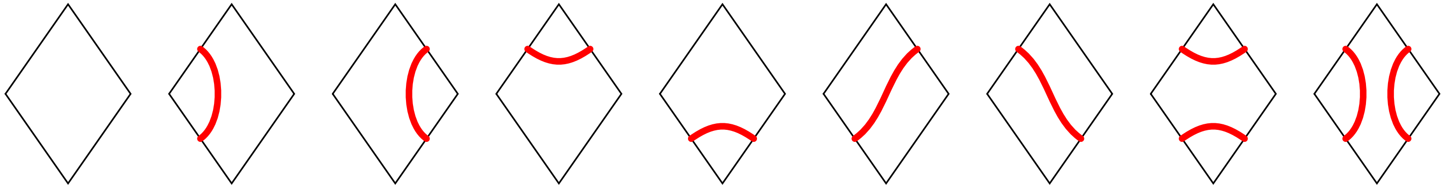


Adding one rhombus to the tiled area changes the shape of the border and thus the corresponding Ψ.

Knowing the weights of the rhombus, we know exactly how Ψ changes:

$$R^{i}_{\alpha,\beta}(u_i,u_{i+1})\,\Psi_{\beta}(\ldots,u_i,u_{i+1},\ldots) = W(u_i,u_{i+1})\,\Psi_{\alpha}(\ldots,u_{i+1},u_i,\ldots)$$

R is the action of the added rhombus on the link pattern,
W is a normalization.



$$R^i_{\alpha,\beta}(u_i,u_{i+1})\,\Psi_\beta(\ldots,u_i,u_{i+1},\ldots)=W(u_i,u_{i+1})\,\Psi_\alpha(\ldots,u_{i+1},u_i,\ldots)$$

Consider the case that in the link pattern α, the positions i and $i{+}1$ are both occupied, and not connected.

Then there is only the right-most diagram in R can contribute.

Therefore for this link pattern the equation reads:

$$R(u_i,u_{i+1})\,\Psi_\alpha(\ldots,u_i,u_{i+1},\ldots)=W(u_i,u_{i+1})\,\Psi_\alpha(\ldots,u_{i+1},u_i,\ldots)$$

This results in:

$$(u_i q - u_{i+1}q^{-1})\,\Psi_\alpha(u_i,u_{i+1}) = (u_{i+1}q - u_i q^{-1})\,\Psi_\alpha(u_{i+1},u_i)$$

with $q^3=-1$.

For two occupied consecutive edges, not mutually connected, the function Ψ contains the factor $(u_{i+1}\, q - u_i\, q^{-1})$ and is otherwise symmetric.

By induction: for a sequence of occupied consecutive edges $n \ldots m$, not mutually connected, the Ψ contains the factor

$$\prod_{i=n}^{m} \prod_{j=i+1}^{m} (q\, u_j - q^{-1}\, u_i)$$

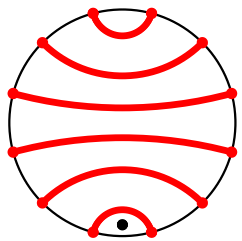
and is otherwise symmetric.

For the most nested element of size $L = 2n$ we thus have the factor

$$\prod_{i=1}^{n} \prod_{j=i+1}^{n} (q\, u_j - q^{-1}\, u_i)$$

We take this for the complete expression.

Can we use this to obtain the other elements?

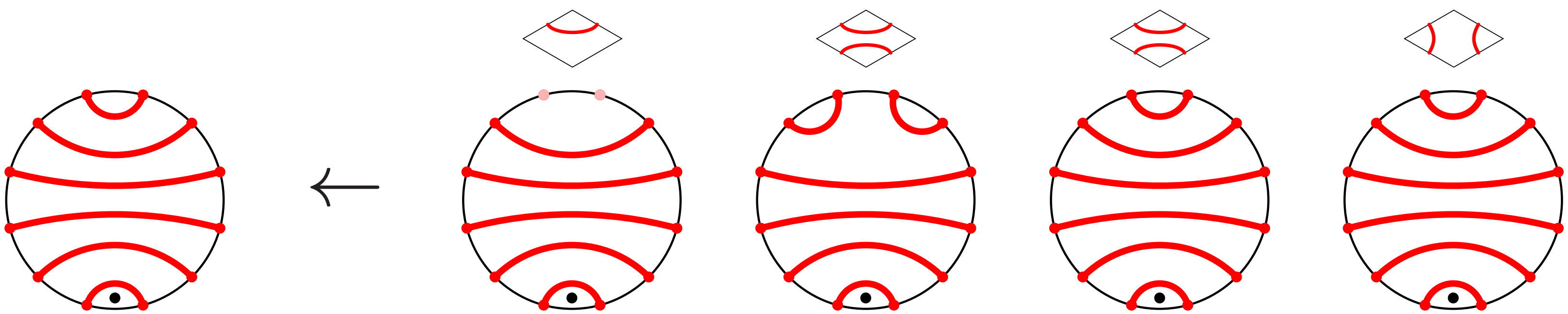


For the most nested element of size $L = 2n$ we thus have the factor

$$\prod_{i=1}^{n} \prod_{j=i+1}^{n} (q\, u_j - q^{-1}\, u_i)$$

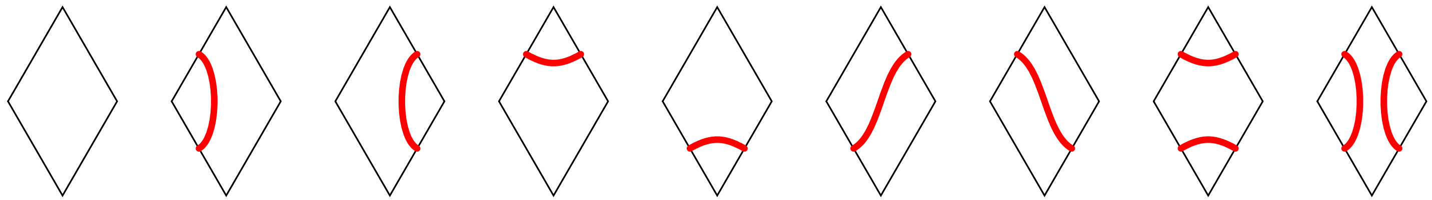
We take this for the complete expression.

Not directly.



In this equations there are **two** new elements.

But the R operator has a pleasant property:
When the corresponding rhombus has an angle of $\pi/3$, the weights are the same, except the last one which vanishes:



The rhombus can be seen as the product of two triangles.

When two consecutive parameters of the border of a tiled area are related as $uq, u/q$, we may presume this rhombus in between, and reduce the border by one edge. Consequently:

$$\Psi_L(\ldots, u\,q, u/q, \ldots) = F_L\,\Psi_{L-1}(\ldots, u, \ldots)$$

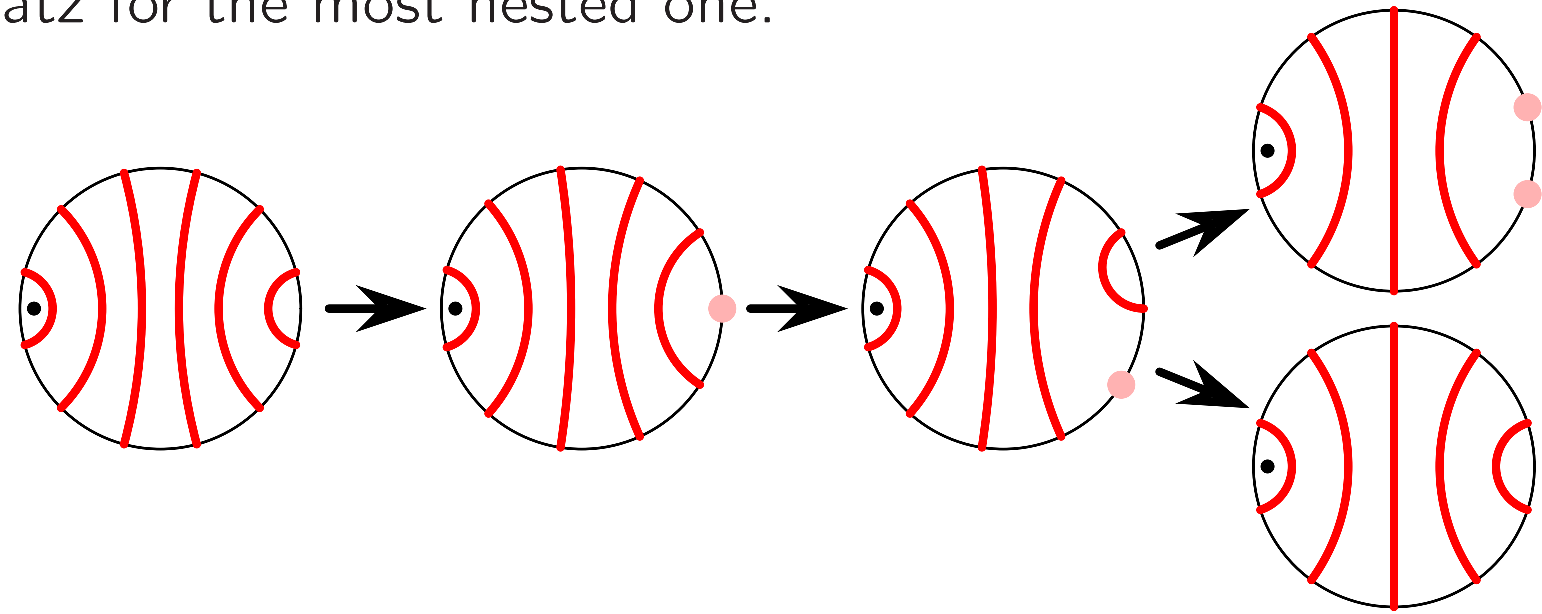
for some F_L, and with the link pattern adjusted according to above triangles.

In order to separate the two triangles we need a variation of the Yang-Baxter equation:

which indeed is satisfied by the weights.

From its shape in terms of vertices, we call it the Yen-Baxter equation.

Using this recursion we can obtain all elements of Ψ from the ansatz for the most nested one.



The lower branch gives the factor F_L, and the upper branch gives a new element.

This scheme recursively produces all elements from the most nested ones.

The sum of all elements and the empty element are symmetric functions of the parameters. And they also satisfy the recursion in size.

They could be recognized as:

$$\Psi_{\mathsf{s},L} = \det_{0\le i,j<L} E_{3i-2j}$$

Here E_n is the elementary symmetric function of degree n

$$E_n = \sum_{k_1<k_2<\cdots<k_n\le L} \prod_{i=1}^{n} u_{k_i}$$

with the property

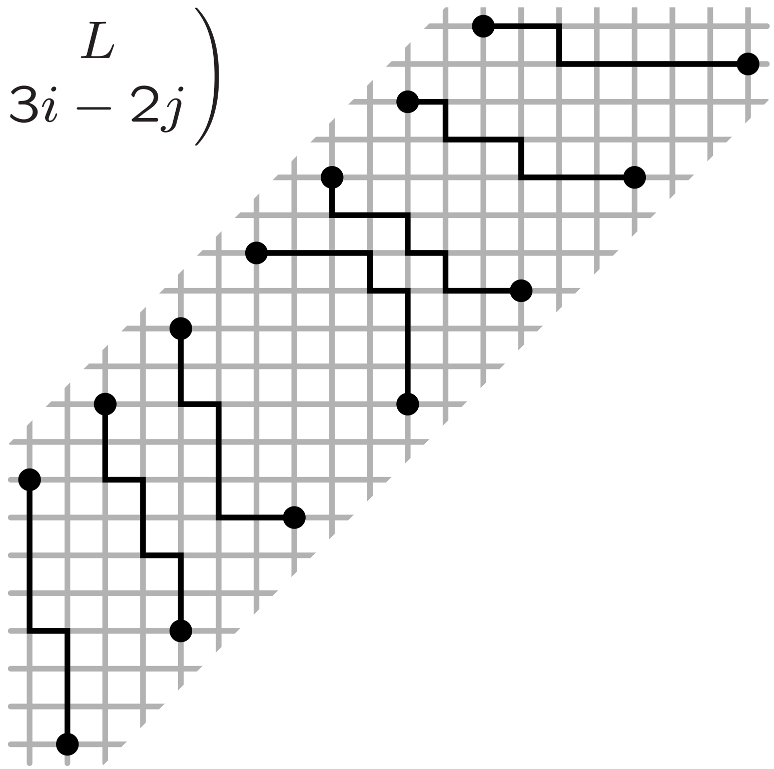
$$E_n(\ldots,u\,q,u/q,\ldots) = E_n(\ldots,u\,\ldots) + u\,E_{n-1}(\ldots,u\,\ldots),$$

which is not difficult to prove.

In order to see a connection to combinatorics we take the homogeneous case, say all $u_j = 1$.

$$\Psi_{\mathsf{E},L} = \det_{0 \le i,j < L} \binom{L}{3i - 2j}$$

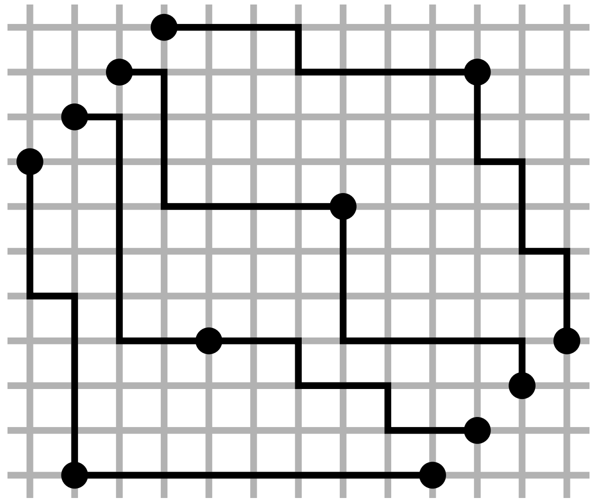
With acknowledgment to Mireille Bousquet Mélou, this can be seen as a Gessel-Viennot determinant for lattice paths over a diagonal strip of the square lattice, between arrays of starting and ending points spaced by 2 and 3 unites respectively.



It may be noted that the corresponding expression for the TL case is:

$$\det_{1\le i,j\le L/2}\binom{L}{L-3i+j}\det_{1\le i,j\le L/2}\binom{L}{L+1-3i+j}$$

Here we have a product of two similar determinants. Again lattice paths acros a diagonal strip.
Now the spacing of the terminals is 1 and 3 units respectively.



Other boundary conditions

With open or reflecting boundaries, the expression for the empty element is similar.

The open boundaries have an additional degree of freedom, say u_0 and u_{L+1}.

The Elementary symmetric functions are now taken of the variables $\{\ldots, u_i, u_i^{-1}, \ldots\}$.

For the SAW or interacting SAW a similar calculation can be done.

The empty element for $L = 2n$ is:

$$\left(\prod_{1\leq i<j\leq 2n} \frac{(u_i\, q + u_j)(u_i/q + u_j)}{(u_i - u_j)}\right) \mathrm{Pf}\left(\frac{(u_i^2 - u_j^2)}{(u_i\, q + u_j)(u_i/q + u_j)}\right)_{1\leq i<j\leq 2n}$$

Here $q = \mathrm{e}^{\mathrm{i}\pi/4}$ for SAW and $q = -\mathrm{e}^{\mathrm{i}\pi/4}$ for I-SAW.

Christian Hagendorf noted that this expression appears in Greg Kuperberg's *Symmetry classes of alternating-sign matrices under one roof* as a factor in the in the x-enumeration of Quarter-Turn symmetric ASM's.

It is remarkable that again Alternating Sign matrices show up.

- The empty (and largest) element of the some cases of the dilute loop model is calculated.

- In the homogeneous case it can be related to lattice paths across a diagonal strip.

- Expressions for other boundary conditions are similar.